

Lecture 14: Encoding Linear Transformations as Matrices. [§2.2 FIS] (1)

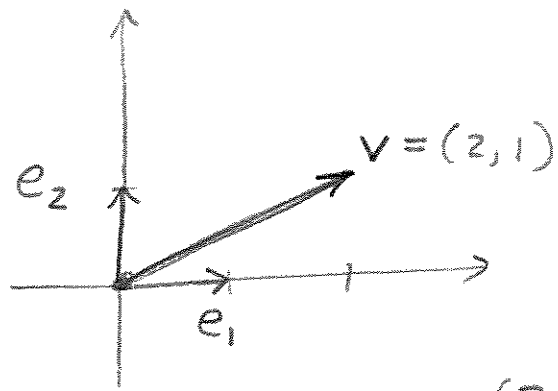
Suppose $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for a vector space V . For any $v \in V$ there are unique scalars such that

$$v = a_1 v_1 + \dots + a_n v_n$$

The coordinate vector of v with respect to β

is

$$[v]_{\beta} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$



Ex: $V = \mathbb{R}^2$

$$\beta = \{e_1, e_2\} \quad v = 2e_1 + 1 \cdot e_2 \Rightarrow [v]_{\beta} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\beta' = \{e_2, e_1\} \quad v = 1 \cdot e_2 + 2e_1 \Rightarrow [v]_{\beta'} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\gamma = \{u_1 = (1, 1), u_2 = (1, -1)\}$$

$$v = \frac{3}{2}u_1 + \frac{1}{2}u_2 \Rightarrow [v]_{\gamma} = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

Note that $[]_{\beta}$ gives a map

$$\begin{array}{ccc} V & \longrightarrow & \mathbb{R}^n \\ v & \longmapsto & [v]_{\beta} \end{array}$$

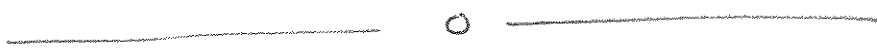
which is

(a) Linear: $[a v_1 + v_2]_{\beta} = a [v_1]_{\beta} + [v_2]_{\beta}$

(b) a 1-1 and onto map, also called a bijective correspondence. The inverse map

is $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \rightarrow a_1 v_1 + \dots + a_n v_n.$

Moral: For a finite dim'd vector space, a choice of basis gives a way to relate it to \mathbb{R}^n .



Now suppose $T: V \rightarrow W$ is linear, and $\beta = \{v_1, \dots, v_n\}$ is a basis for V and $\gamma = \{w_1, \dots, w_m\}$ is a basis for W

From last time, know $\{T(v_1), \dots, T(v_n)\}$ determine T . (3)

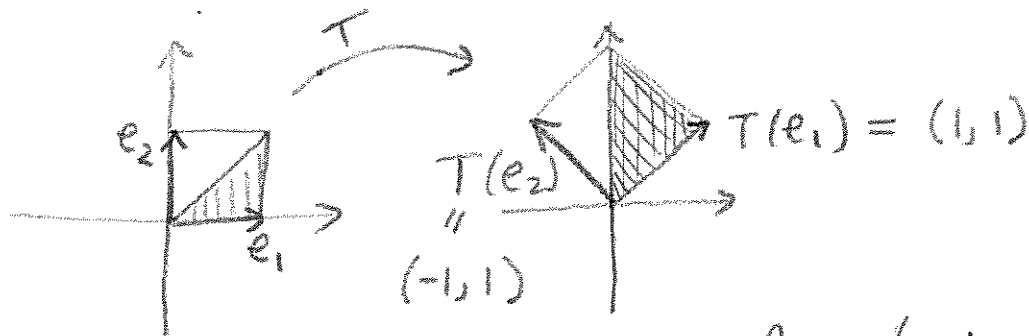
The matrix of T in the bases β and γ is

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} | & & | \\ [T(v_1)]_{\gamma} & \cdots & [T(v_n)]_{\gamma} \\ | & & | \end{pmatrix} \in M_{m \times n}(\mathbb{R})$$

Equivalently, its entries are the unique scalars a_{ij} where

$$T(v_j) = a_{1j}w_1 + a_{2j}w_2 + \cdots + a_{mj}w_m$$

Ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $T(x, y) = (x - y, x + y)$



$$[T]_{\beta}^{\beta} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad [T]_{\beta'}^{\beta} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$[T]_{\beta'}^{\beta'} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad [T]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T(e_1) = e_2 + e_1$$

$$T(e_2) = e_2 - e_1$$

$$T(e_1) = u_1$$

$$T(e_2) = -u_2$$

$$[T]_{\gamma}^{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$T(u_1) = (0, 2) = u_1 - u_2$$

$$T(u_2) = (2, 0) = u_1 + u_2$$

Suppose $A \in M_{m \times n}$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ a column vector of height n . Their product Ax is the column y vector of height m where

$$y_i = A_{i1}x_1 + A_{i2}x_2 + \dots + A_{in}x_n = \sum_{j=1}^n A_{ij}x_j$$

Ex: $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -3 & 4 \end{pmatrix}$ $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $Ax = \begin{pmatrix} 2 \cdot 1 + 1 \cdot 2 + 0 \cdot 3 \\ 0 \cdot 1 - 3 \cdot 2 + 4 \cdot 3 \end{pmatrix}$
 $= \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Relevance: ① $LS(A, b)$ is $A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b$
 \uparrow variables

② $T: V \rightarrow W$ with β a basis for V
 γ a basis for W

⑤

Then

$$[T(v)]_{\gamma} = [T]_{\beta}^{\gamma} [v]_{\beta}$$

[Will justify momentarily, but first let's use.]

Ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ $T(x, y) = (x - y, x + y)$

$$\beta = \{e_1, e_2\}$$

$$v = (2, 1)$$

$$\gamma = \{u_1 = (1, 1), u_2 = (1, -1)\}$$

$$\bullet [T(v)]_{\beta} = [(1, 3)]_{\beta} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$[T]_{\beta}^{\beta} [v]_{\beta} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \checkmark$$

$$\bullet [T(v)]_{\gamma} = [(1, 3)]_{\gamma} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$[T]_{\gamma}^{\gamma} [v]_{\gamma} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \checkmark$$

$$\bullet [T]_{\beta}^{\gamma} [v]_{\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \checkmark$$

Proof that $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$. (6)

$$\beta = \{v_1, \dots, v_n\} \quad \gamma = \{w_1, w_2, \dots, w_m\}$$

If $v = a_1 v_1 + \dots + a_n v_n$ then by linearity have

$$T(v) = a_1 T(v_1) + \dots + a_n T(v_n).$$

Now the map $W \rightarrow \mathbb{R}^m$ is itself linear,
 $w \mapsto [w]_\gamma$

so

$$\begin{aligned} [T(v)]_\gamma &= a_1 [T(v_1)]_\gamma + \dots + a_n [T(v_n)]_\gamma \\ &= \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \dots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ &= [T]_\beta^\gamma [v]_\beta \end{aligned}$$

▣

Thm: Suppose V and W are vector spaces with basis β, γ with $n = \#\beta$ and $m = \#\gamma$. Given $A \in M_{m \times n}(\mathbb{R})$, there is a unique linear transformation $T: V \rightarrow W$ with $[T]_\beta^\gamma = A$.

Pf: Later this week.