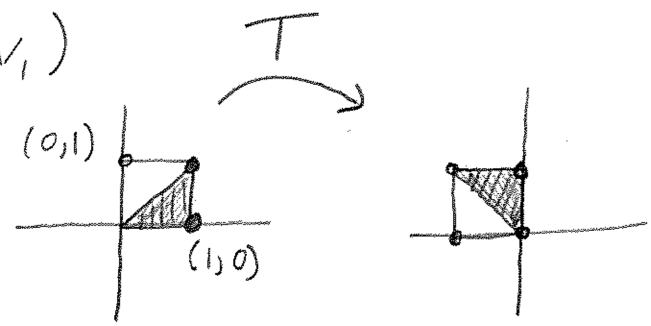
Lecture 13: Linear transformations: The Dimension Thm [FIS § 2.1]

Recall: V, W vector spaces. A function $T: V \to W$ is a linear transformation (or just linear) if for all $v_1, v_2 \in V$ and $a \in \mathbb{R}$ we have

a) $T(v_1 + v_2) = T(v_1) + T(v_2)$

b) $T(a v_1) = a T(v_1)$



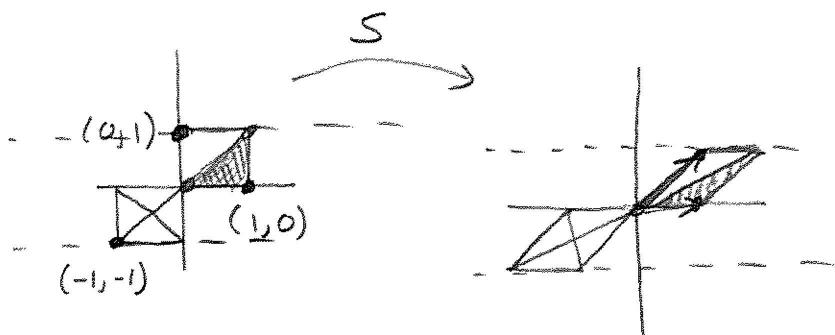
Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x, y) = (-y, x)$$

rotation by $\pi/2$ counterclockwise.

$S: \mathbb{R}^2 \to \mathbb{R}^2$

$$(x, y) \longrightarrow (x+y, y)$$



Ex: $T: \mathcal{C}([0,1]) \longrightarrow \mathbb{R}$

$$\mathcal{C}([0,1]) = \{ f \in \mathcal{F}([0,1], \mathbb{R}) \mid f \text{ is continuous} \}$$

$$T(f) = \int_0^1 f(x)\,dx. \qquad T(x) = \tfrac{1}{2} \quad T(x^2) = 1/3$$

That this T is linear is just the basic props of definite integrals:

$$\int_0^1 c\,f(x) + g(x)\,dx = c\int_0^1 f(x)\,dx + \int_0^1 g(x)\,dx.$$

[Heading toward the Dim Thm...]

<u>Thm</u>: Suppose $T: V \longrightarrow W$ is linear. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V then T is determined by its values on β. Moreover

$$\underbrace{R(T)}_{\text{range of } T} = \{T(v) \mid v \in V\} = \operatorname{span}(\{T(v_1), \ldots, T(v_n)\})$$

<u>Proof</u>: Suppose we know $T(v_1), \ldots, T(v_n)$. Given $v \in V$ there are unique scalars such that

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

Repeatedly using props a) and b) for linear transformations, we get

$$T(v) = a_1 T(v_1) + a_2 T(v_2) + \cdots + a_n T(v_n).$$

So the $T(v_i)$ determine T. Also we've learned that $R(T) \subseteq \text{span}(\{T(v_i)\})$

As $T(v_i) \in R(T)$ and $R(T)$ is a subspace, have $R(T) \supseteq \text{span}(\{T(v_i)\})$. So $R(T) = \text{span}(\{T(v_i)\})$ as claimed. $\blacksquare$

Recall $\mathcal{N}(T: V \to W) = \{ v \in V \mid T(v) = 0 \}$ is also a subspace.

<u>Dimension Thm</u>: Suppose $T: V \to W$ is linear. If V is finite dim'l, then

$$\underbrace{\dim(\mathcal{N}(T))}_{\text{nullity of } T} + \underbrace{\dim(R(T))}_{\text{rank of } T} = \dim V$$

[Last time, gave some examples. Here are more.]

<u>Ex</u>: $T: \mathbb{R}^{a+b} \longrightarrow \mathbb{R}^{a+c}$

$$(x_1, \dots, x_n) \quad (x_1, \dots, x_a, \underbrace{0, \dots 0}_{c \text{ zeros}})$$

So $R(T) = \{ (x_1, \dots, x_a, 0, \dots, 0) \}$.

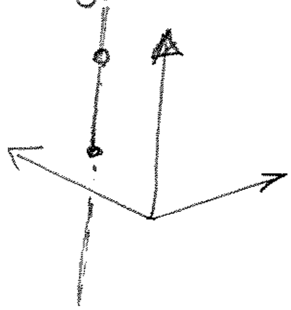
$$\mathcal{N}(T) = \{(0,\dots,0,x_{a+1},\dots,x_{a+b}) \mid x_{a+1},\dots,x_{a+b} \in \mathbb{R}\}$$

So

$$\text{nullity} + \text{rank} = b + a = \dim \mathbb{R}^{a+b}.$$

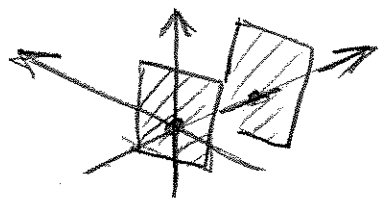
Ex: $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$(x,y,z) \longmapsto (x,y)$



$\mathbb{R}^3 \longrightarrow \mathbb{R}$

$(x,y,z) \longrightarrow x$



Pf: [Idea: In the right coordinates (= bases) any linear transformation looks like these exs.]

Let β' be a basis for $\mathcal{N}(T)$. By cor of the Repl. Thm can enlarge this to a basis β of V, say $\beta = \{v_1,\dots,v_a,\underbrace{v_{a+1},\dots,v_{a+b}}_{\beta'}\}$

It's enough to show that

$$\gamma = \{T(v_1),\dots,T(v_a)\}$$

is linearly independent as then by last

theorem it is a basis of $R(T)$ and so

$$\text{nullity} + \text{rank} = b + \#\gamma = b + a = \#\beta = \dim V.$$

Suppose have c_i with $c_1 T(v_1) + \cdots + c_a T(v_a) = 0$

By linearity of T, have is equal to

$$T(c_1 v_1 + \cdots + c_a v_a).$$

So $w = c_1 v_1 + \cdots + c_a v_a \in \mathcal{N}(T)$, and so

it is a linear comb. of β' with

$$w = c_{a+1} v_{a+1} + \cdots + c_{a+b} v_{a+b}$$

By linear indep of β, we must have all $c_i = 0$. So γ is linearly independent as needed to prove the theorem. $\blacksquare$