Lecture 29: Markov chains (§5.3)

A tale of two cities: Champaign vs. Urbana

Population in 2000: 30,000 70,000

Closed system with no one moving to or from the area. After one year, 20% of Urbanites have moved to Champaign, and 10% of Champ. residents have moved to Urbana.

Assuming this pattern continues, what are the cities populations in 2010? in 2020? in the long term.

Population in 2001:

$$\text{Champaign} = 0.9 \cdot 30k + 0.2 \cdot 70k = 41k$$
$$\text{Urbana} = 0.1 \; 30k + 0.8 \cdot 70k = 59k$$

If we record the population in a vector $\begin{pmatrix} \text{Champ.} \\ \text{Urb} \end{pmatrix}$ so that $P_{2000} = \begin{pmatrix} 30k \\ 70k \end{pmatrix}$. If we set

$A = \begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}$ then note $P_{2001} = A P_{2000}$.

This is not specific to our initial population. For example if $p'_{2000} = \begin{pmatrix} 50k \\ 50k \end{pmatrix}$ then again $p'_{2001} = A\, p'_{2000}$

$= \begin{pmatrix} 55k \\ 45k \end{pmatrix}$.

What about 2002? Assume <u>same</u> portions of the <u>new</u> populations switch cities (ahistorical / no memory). So $p_{2002} = A p_{2001} = \begin{pmatrix} 48.7k \\ 51.3k \end{pmatrix}$

and in general $p_{2000+n} = A^n p_{2000}$.

In particular, $p_{2010} = A^{10} p_{2000} = \begin{pmatrix} 65,631 \\ 34,369 \end{pmatrix}$

$$p_{2020} = A^{20} p_{2000} = \begin{pmatrix} 66,637 \\ 33,362 \end{pmatrix}$$

<u>Q</u>: What will happen long term?

<u>Guess</u>: 2/3 of pop in Champaign and 1/3 in Urbana.

[Idea: Use diagonalization to prove this]

Turns out that eigenvalues are 1 and 7/10, with

$$E_1 = \text{span}\{(2,1)\}$$
$$E_{7/10} = \text{span}\{(1,-1)\}$$

So if $Q = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$ then $Q^{-1}AQ = D = \begin{pmatrix} 1 & 0 \\ 0 & 7/10 \end{pmatrix}$.

and so $A = QDQ^{-1}$! Hence

$$P_{2000+n} = A^n P_{2000} = QD^nQ^{-1} P_{2000}$$
$$= Q\begin{pmatrix} 1^n & 0 \\ 0 & (7/10)^n \end{pmatrix} Q^{-1} P_{2000}$$

As discussed in the text, can take calculus-style limits and manipulate as expected:

$$\lim_{n\to\infty} P_{2000+n} = Q\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} Q^{-1} P_{2000}$$

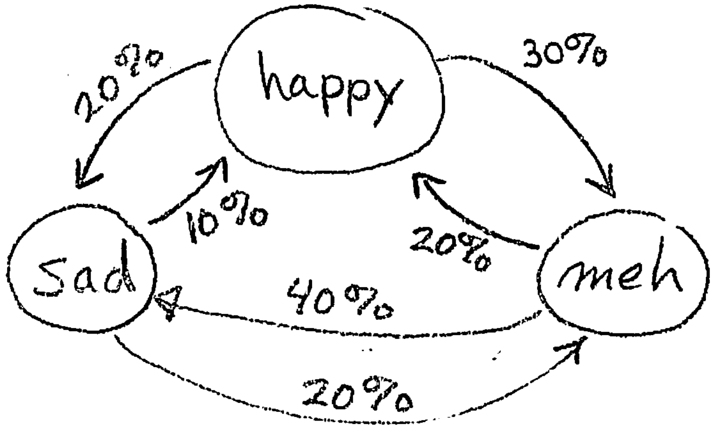
$$= \begin{pmatrix} 2/3 & 2/3 \\ 1/3 & 1/3 \end{pmatrix}\begin{pmatrix} 30k \\ 70k \end{pmatrix} = 100k \cdot \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

<u>Important Note</u>: In the long term, only the total population in year 2000

matters. Specifically, get same limiting populations if $P'_{2000} = \begin{pmatrix} 50k \\ 50k \end{pmatrix}$.

A tale of three mental states:

Prof N:



State changes once an hour, at random with the probabilities shown. Suppose he is happy at time 0, what is the prob that he is also happy at time 10? or at time 100?

Set $P_n = \begin{pmatrix} \text{prob happy at time } n \\ \text{prob meh at time } n \\ \text{prob sad at time } n \end{pmatrix}$ so $P_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

and consider

$$A = \begin{array}{c} \text{current state} \\ \begin{array}{ccc} h & m & s \end{array} \\ \begin{pmatrix} 0.5 & 0.2 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.7 \end{pmatrix} \end{array} \begin{array}{l} h \\ m \\ s \end{array} \quad \text{state after one hour}$$

Then $p_n = A p_{n-1} = A^n p_0$. Again, look at eigenstuff.

$\lambda_1 = 1 \qquad E_1 = \text{span}(\{u_1 = (10, 13, 24)\})$

$\lambda_2 \approx 0.159 \qquad E_{\lambda_2} = \text{span}(\{u_2\})$

$\lambda_3 \approx 0.441 \qquad E_{\lambda_3} = \text{span}(\{u_3\})$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

So set $Q = \begin{pmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}$ so that $A = QDQ^{-1}$

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} A^n p_0 = \lim_{n\to\infty} Q D^n Q^{-1} p_0$$

$$= Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} Q^{-1} p_0$$

$$= \begin{pmatrix} | & | & | \\ \bar{u}_1 & \bar{u}_1 & \bar{u}_1 \\ | & | & | \end{pmatrix} p_0 = \bar{u}_1 \quad \text{where } \bar{u}_1 = \frac{1}{47}\begin{pmatrix} 10 \\ 13 \\ 24 \end{pmatrix}$$

Skipping messy calculation!

Def: A <u>probability vector</u> $v = (a_1, \ldots, a_n) \in \mathbb{R}^n$ is one where all $a_i \geq 0$ and $\sum a_i = 1$.

A <u>transition matrix</u> is an $A \in M_{n\times n}(\mathbb{R})$ where all $A_{ij} \geq 0$ and each <u>column</u> sums to 1.

Thm: Suppose $A \in M_{n \times n}(\mathbb{R})$ where every entry is positive. Then

a) 1 is an eigenvalue of A and $\dim E_1 = 1$.

b) All other eigenvalues λ have $|\lambda| < 1$.

c) There is a prob. vector v in E_1, and for any prob vector $w \in \mathbb{R}^n$ have

$$\lim_{n \to \infty} A^n w = v.$$