Lecture 36: Normal and self-adjoint operators (§6.4)

Last time: Suppose T is a linear op on a finite dim'l inner product space V.

The adjoint T^* of T is the linear op on V where $\langle T(x), y\rangle = \langle x, T^*(y)\rangle$ for all $x, y \in V$.

Thm: If β is orthonormal, then $[T^*]_\beta = ([T]_\beta)^*$.

[Today: some special kinds of ops with orthogonal eigenvectors. Will use theory developed to show:]

Goal: Suppose $A \in M_{n\times n}(\mathbb{R})$ or $M_{n\times n}(\mathbb{C})$.
If $A^* = A$, then A is diagonalizable.

Ex: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 6 & 7 \\ 3 & 6 & 8 & 9 \\ 4 & 7 & 9 & 10 \end{pmatrix}$ is diagonalizable.

Def: A linear op T on an inner product space V is normal if $T \circ T^* = T^* \circ T$.
A square matrix is normal if $AA^* = A^*A$.

Ex: $V = (\mathbb{R}^2, \text{dot})$ $\quad T = \begin{matrix}\text{rotation} \\ \text{by } \theta\end{matrix} = L_A$

where $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then $T^* = L_{A^*}$

where $A^* = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and $AA^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = A^*A$.

[Recall from HW the condition that $AA^t = I$ is called orthogonal.]

Ex: Suppose A is a matrix with $A^* = -A$,

for example $A = \begin{pmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{pmatrix}$. Then $A^*A = AA^* = -A^2$

Thm: Suppose T is a linear operator on a finite dim'l inner product space V.

If T has an orthonormal basis β of eigenvectors, then T is normal.

Proof: It suffices to show $[T \circ T^*]_\beta = [T^* \circ T]_\beta$.

Now $[T]_\beta$ is diagonal, and hence so is

$[T^*]_\beta = ([T]_\beta)^*$. As diagonal matrices commute, have $[T]_\beta [T^*]_\beta = [T^*]_\beta [T]_\beta$ as needed. $\blacksquare$

<u>Q</u>: Must a normal operator T have an orthonormal basis of eigenvectors?

<u>A</u>. If the field of scalars is $\mathbb{C}$, then yes (Theorem 6.16 in text). Over $\mathbb{R}$, the rotation example above shows the answer is no.

[Over $\mathbb{R}$, want $T = T^*$ to be able to diagonalize...]

<u>Thm</u>: For a normal op T on V the following hold:

a) $\|T(v)\| = \|T^*(v)\|$ for all $v \in V$.

b) $T + cI_V$ is normal for all scalars c.

c) If v is an eigenvector for T with eigenvalue λ, then v is an eigenvector for T^* with eigenvalue $\overline{\lambda}$.

d) If v_1, v_2 are eigenvectors for T cor to distinct eigenvalues λ_1, λ_2, then v_1 and v_2 are orthogonal.

<u>Proof</u>: a)
$$\|T(v)\|^2 = \langle T(v), T(v)\rangle = \langle v, T^* \circ T(v)\rangle$$
$$= \langle v, T \circ T^*(v)\rangle = \langle T^*(v), T^*(v)\rangle$$
$$= \|T^*(v)\|^2.$$

b)
$$(T + cI_V) \circ (T + cI_V)^* = (T + cI_V) \circ (T^* + \bar{c} I_V)$$
$$= T \circ T^* + \bar{c}T + cT^* + |c|^2 I_V$$
$$= T^* \circ T + \bar{c}T + cT^* + |c|^2 I_V$$
$$= (T^* + \bar{c} I_V)(T + cI_V) = (T + cI_V)^*(T + cI_V).$$

c)
$$v \in E_\lambda(T) \iff v \in \mathcal{N}(T - \lambda I)$$
$$\iff \|(T - \lambda I)v\| = 0$$
$$\iff \|\underbrace{(T - \lambda I)^*}_{(T^* - \bar{\lambda} I)} v\| = 0 \quad \text{(by (b) and (a))}$$
$$\iff v \in \mathcal{N}(T^* - \bar{\lambda} I)$$
$$\iff v \in E_{\bar{\lambda}}(T^*).$$

d) $\lambda_1 \langle v_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \langle T(v_1), v_2 \rangle$

$= \langle v_1, T^*(v_2) \rangle = \langle v_1, \overline{\lambda}_2 v_2 \rangle$

$= \lambda_2 \langle v_1, v_2 \rangle$

Thus if $\lambda_1 \neq \lambda_2$ must have $\langle v_1, v_2 \rangle = 0$. ▨

<u>Def</u>: A linear op T on an inner product space V is <u>self-adjoint</u> if $T = T^*$. Similarly, a matrix is <u>self-adjoint</u> if $A = A^*$.

<u>Note</u>: A self-adjoint operator is normal since $T^* \circ T = T^2 = T \circ T^*$.

<u>Thm</u>: If T is self-adjoint, then any eigenvalue λ of T is real.

<u>Proof</u>: Let v be an eigenvector with eigenvalue λ. By (c), v is also an eigenvector of T^* with eigenvalue $\overline{\lambda}$.

As $T^* = T$, we get $T(v) = \lambda v$ and $\bar{\lambda} v$
which implies $\lambda = \bar{\lambda}$ as $v \neq 0$. ▨

<u>Lemma</u>: Suppose T is a self-adjoint op of a finite dim'l inner product space V <u>over $\mathbb{R}$</u>.
Then T has at least one eigenvector.

<u>Proof</u>: Let β be an orthonormal basis for V, and set $A = [T]_\beta$. As $T = T^*$ we have $A = A^* = A^t$, that is A is symmetric.
It suffices to prove that the char poly $f(t)$ of A has a real root. Consider $L_A : \mathbb{C}^n \to \mathbb{C}^n$ where $\mathbb{C}^n$ has the usual inner product. Now L_A is self-adjoint since $(L_A)^* = L_{A^*}$. As $f(t)$ splits completely over $\mathbb{C}$ and by last theorem we know all the roots are real. So $f(t)$ splits completely over $\mathbb{R}$ and hence T has an eigenvector. ▨