

Lecture 32: More on inner products (§ 6.1 and 6.2) ①

Last time: V a vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . An inner product is a $\langle \cdot, \cdot \rangle: V^2 \rightarrow \mathbb{F}$ satisfying

a) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

b) $\langle cx, y \rangle = c \langle x, y \rangle$

c) $\langle y, x \rangle = \overline{\langle x, y \rangle}$

d) $\langle x, x \rangle$ is in $\mathbb{R}_{>0}$ for $x \neq 0$.

for all $x, y, z \in V$ and $c \in \mathbb{F}$.

From $\langle \cdot, \cdot \rangle$, we define $\|x\| = \sqrt{\langle x, x \rangle}$ for $x \in V$ and call it the norm/length of x .

Thm: Suppose V is an inner product space.

For all $x, y \in V$ and scalar c one has:

a) $\|cx\| = |c| \|x\|$

b) $\|x\| \iff x=0$

c) $|\langle x, y \rangle| \leq \|x\| \|y\|$ (Cauchy Schwarz)

$$d) \|x+y\| \leq \|x\| + \|y\| \quad (\Delta\text{-inequality}) \quad (2)$$

Proof: Parts (a) and (b) are easy; will do (c) and (d) when the field of scalars is \mathbb{R} .

c) If $y = 0$ get $0 \leq 0$ as needed, so assume $y \neq 0$.

Notice that if we scale y by $c \in \mathbb{R}$, then both sides of (c) change by $|c|$ as per part (a).

Replacing y with $\frac{1}{\|y\|} y$ we can thus assume that

$\|y\| = 1$. Now

$$\begin{aligned} 0 \leq \|x - \langle x, y \rangle y\|^2 &= \langle x, x \rangle - 2\langle x, y \rangle^2 + \langle x, y \rangle^2 \overbrace{\langle y, y \rangle}^1 \\ &= \|x\|^2 - \langle x, y \rangle^2 \end{aligned}$$

and so $|\langle x, y \rangle| \leq \|x\|$ as needed.

$$\begin{aligned} d) \|x+y\|^2 &= \langle x+y, x+y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\ &\leq \|x\|^2 + 2|\langle x, y \rangle| + \|y\|^2 \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad (\text{by (c)}), \\ &= (\|x\| + \|y\|)^2 \end{aligned}$$



(3)

Def: Suppose V is an inner product space.

Vectors $x, y \in V$ are orthogonal / perpendicular when

$\langle x, y \rangle = 0$. A subset $S \subseteq V$ is orthogonal when each pair of distinct vectors is orthogonal. A

vector $x \in V$ is unit when $\|x\| = 1$. Finally,

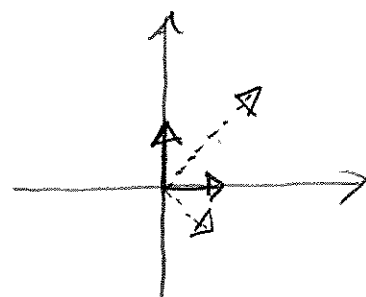
a subset of V is orthonormal when it is orthogonal and consists entirely of unit vectors.

Ex: \mathbb{R}^2 , dot product

• $S = \{e_1, e_2\}$ is orthonormal

• $S = \{2e_1 + 2e_2, e_1 - e_2\}$ is orthogonal.

• $S = \{e_1, e_1 + e_2\}$ is not orthogonal.



• $S = \left\{ \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}$
is orthonormal

Def: An orthonormal basis for an inner product space V is a basis which is orthonormal.

Thm: An finite dim'l inner product space has an orthonormal basis.

[Will prove this next time...]

(4)

Thm: Suppose $S = \{v_1, v_2, \dots, v_k\}$ is an orthonormal subset of an inner product space V .

If $y \in \text{span}(S)$, then $y = \sum_{i=1}^k \langle y, v_i \rangle v_i$.

Ex: (\mathbb{R}^3 , dot product) $S = \{e_1, e_2, e_3\}$

$$y = (4, 3, -2) = 4e_1 + 3e_2 - 2e_3$$

$$\langle y, e_1 \rangle = 4 \quad \langle y, e_2 \rangle = 3 \quad \langle y, e_3 \rangle = -2.$$

Proof: As $y \in \text{span}(S)$, there are scalars a_i with $y = \sum_{i=1}^k a_i v_i$. Then $\langle y, v_j \rangle = \sum_{i=1}^k \langle a_i v_i, v_j \rangle$

$$= \sum_{i=1}^k a_i \langle v_i, v_j \rangle = a_j \text{ for all } 1 \leq j \leq k.$$

$\underbrace{\hspace{10em}}_{= 0 \text{ when } i \neq j}$
 $= \|v_j\|^2 = 1 \text{ when } i = j$

Thus $\sum_{i=1}^k \langle y, v_i \rangle v_i = \sum_{i=1}^k a_i v_i = y$ as needed \square

Cor: Suppose $S = \{v_1, \dots, v_k\}$ is orthogonal with

all $v_i \neq 0$. If $y \in \text{span}(S)$ then $y = \sum_{i=1}^k \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$

Proof: Set $\bar{v}_i = \frac{v_i}{\|v_i\|}$ so that $\bar{S} = \{\bar{v}_1, \dots, \bar{v}_k\}$ (5)

is orthonormal. As $\text{span}(\bar{S}) = \text{span}(S)$,

given $y \in \text{span}(S)$ we have

$$y = \sum_{i=1}^k \underbrace{\langle y, \bar{v}_i \rangle}_{\frac{1}{\|v_i\|} \langle y, v_i \rangle} \bar{v}_i = \sum_{i=1}^k \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i.$$



Cor: Suppose $S \subseteq V$ is orthogonal and $0 \notin S$.

Then S is linearly independent.

Pf: Since scaling vectors by non-zero amounts does not change dependence, it suffices

to show that any orthonormal v_1, \dots, v_k are linearly indep. If $\sum_{i=1}^k a_i v_i = 0$, then

$$\begin{aligned} \text{as before we have } 0 &= \langle v_j, 0 \rangle = \langle v_j, \sum_{i=1}^k a_i v_i \rangle \\ &= a_j \end{aligned}$$

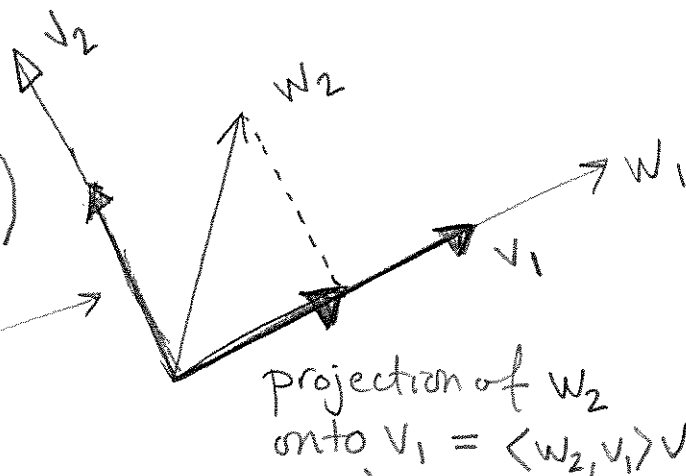
for each j . Thus v_1, \dots, v_k is linearly indep as needed.



Gram-Schmidt Process: Suppose $\{w_1, w_2, \dots, w_n\}$ is a linearly independent subset of V . Set

$$v_1 = \text{unit}(w_1) = \frac{w_1}{\|w_1\|}$$

$$v_2 = \text{unit}(w_2 - \langle w_2, v_1 \rangle v_1)$$



$$v_3 = \text{unit}(w_3 - (\langle w_3, v_1 \rangle v_1 + \langle w_3, v_2 \rangle v_2))$$

and in general

$$v_k = \text{unit}\left(w_k - \sum_{i=1}^{k-1} \langle w_k, v_i \rangle v_i\right)$$

Underlying geometry for v_3 :

