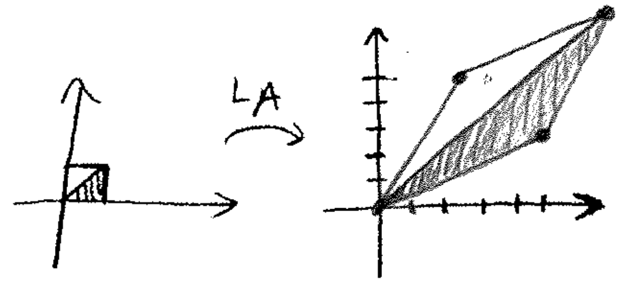
Lecture 19: Changing coordinates [§2.5 of FIS]

Key technique: Simplifying by changing coordinates

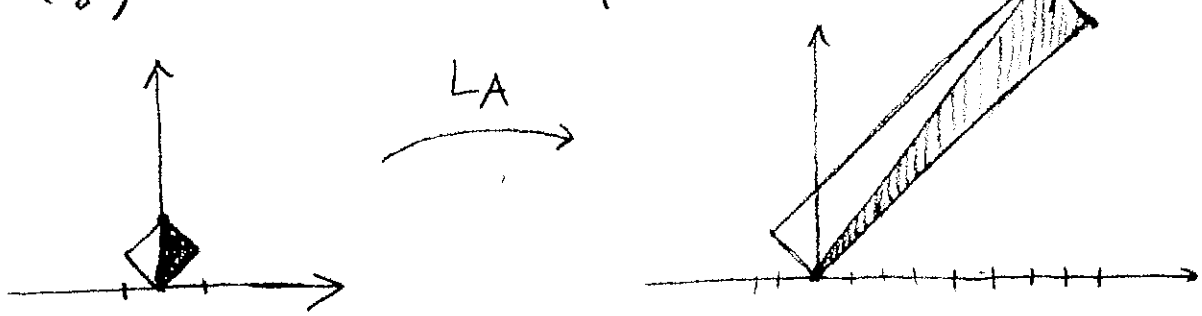
- polar / spherical coordinates in multivariable integration.
- Diagonalization of matrices.

Ex: $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$ $\quad L_A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



$\beta' = \{ v_1 = (1,1) \, , \, v_2 = (-1,1) \}$

$Av_1 = \begin{pmatrix} 8 \\ 8 \end{pmatrix} = 8 v_1 \qquad A v_2 = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = 2 v_2 \qquad [L_A]_{\beta'} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$



Q: Given $T : V \longrightarrow W$ with bases β, β' for V and γ, γ' for W. If we know $[T]_\beta^\gamma$ how do we find $[T]_{\beta'}^{\gamma'}$?

First step: Suppose $v \in V$. How can we compute $[v]_{\beta'}$ from $[v]_\beta$?

Ans: Use identity transformation $I_V : V \to V$, $v \mapsto v$

via

$$[v]_{\beta'} = [I_V(v)]_{\beta'} = [I_V]_\beta^{\beta'} [v]_\beta$$

Note that the columns of $[I_V]_\beta^{\beta'}$ are just the vectors of β written in terms of β'.

Ex: $V = \mathbb{R}^2$ $\quad \beta = \{e_1, e_2\}$ $\quad \beta' = \{v_1 = (1,1),\ v_2 = (-1,1)\}$

$$e_1 = \tfrac{1}{2}(v_1 - v_2) \qquad [I_{\mathbb{R}^2}]_\beta^{\beta'} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$$

$$e_2 = \tfrac{1}{2}(v_1 + v_2)$$

$$[(8,6)]_{\beta'} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$$

Check: $7v_1 - v_2 = (7,7) + (1,-1) = (8,6)$.

Suppose know β' in terms of β, as we did in this example. Then we know $[I_V]_{\beta'}^{\beta}$.

Thm: $[I_V]_\beta^{\beta'} = \left([I_V]_{\beta'}^{\beta}\right)^{-1}$

Pf: $[I_V]_\beta^{\beta'}[I_V]_{\beta'}^{\beta} = [\underbrace{I_V \circ I_V}_{=I_V}]_{\beta'}^{\beta'} = I_n$

and the same for the other product. ▨

Ex: $[I_{\mathbb{R}^2}]_{\beta'}^{\beta} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Useful fact: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\underbrace{ad-bc}_{\det A} \neq 0$ in which case

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$[I_{\mathbb{R}^2}]_{\beta}^{\beta'} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

which is what we got earlier.

Back to original question: $T: \underset{\beta,\beta'}{V} \longrightarrow \underset{\gamma,\gamma'}{W}$

$$[T]_{\beta'}^{\gamma'} = [I_W \circ T \circ I_V]_{\beta'}^{\gamma'}$$

$$= [I_W]_{\gamma}^{\gamma'}[T]_{\beta}^{\gamma}[I_V]_{\beta'}^{\beta}$$

When $V = W$, then

$$[T]_{\beta'} = [I_V]_{\beta}^{\beta'} [T]_{\beta} [I_V]_{\beta'}^{\beta}$$

$$= Q^{-1} [T]_{\beta} Q \quad \text{where } Q = [I_V]_{\beta'}^{\beta}$$

<u>Ex</u>: $V = \mathbb{R}^2 \quad T = L_A$ with $A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$

$\beta = \{e_1, e_2\} \quad \beta' = \{v_1 = (1,1),\ v_2 = (-1,1)\}$

$Q = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$[T]_{\beta'} = Q^{-1} A Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{\begin{pmatrix} 8 & -2 \\ 8 & 2 \end{pmatrix}} = \frac{1}{2}\begin{pmatrix} 16 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 0 & 2 \end{pmatrix}$$

which matches what we calculated earier.

<u>Def</u>: Matrices A, B in $M_{n \times n}(\mathbb{R})$ are <u>similar</u> when there exists an invertible $Q \in M_{n \times n}(\mathbb{R})$ with $B = Q^{-1} A Q$

<u>Note</u>: This is an equivalence relation.

- $B = Q^{-1}AQ \Rightarrow QBQ^{-1} = (QQ^{-1})A(QQ^{-1})$
$$= A$$
and $(Q^{-1})^{-1} = Q$.

- A similar to B, B similar to C

$B = Q^{-1}AQ \qquad C = R^{-1}BR$

$\Rightarrow C = R^{-1}Q^{-1}AQR \underset{\uparrow}{=} (QR)^{-1}A(QR)$

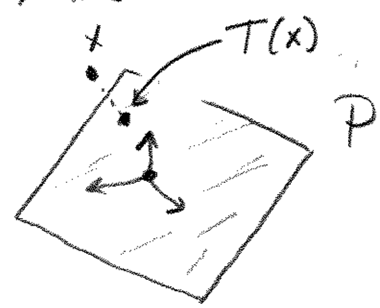
HW!

So A is similar to C.

<u>Ex</u>: Building T using change of coordinates

$V = \mathbb{R}^3 \quad P = \{x+y+z=0\} = \text{span}\{\underset{v_1}{(1,-1,0)}, \underset{v_2}{(0,1,-1)}\}$

$T: \mathbb{R}^3 \to \mathbb{R}^3$ which is orthogonal projection onto P.

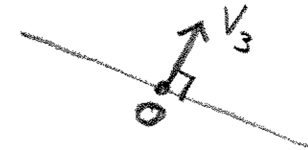


$T(x)$ = closest point on P to x.

<u>Goal</u>: $[T]_\beta$ where $\beta = \{e_1, e_2, e_3\}$ is the standard basis.

Set $\beta' = \{ v_1, v_2, v_3 = (1,1,1) \}$

$$T(v_1) = v_1 \qquad T(v_2) = v_2 \qquad \underbrace{T(v_3) = 0}_{\text{as } v_3 \text{ is normal to } P}$$



So

$$[T]_{\beta'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[T]_{\beta} = \underbrace{[I_{\mathbb{R}^3}]_{\beta'}^{\beta}}_{=A} \, [T]_{\beta'} \, \underbrace{[I_{\mathbb{R}^3}]_{\beta}^{\beta'}}_{=A^{-1}}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \qquad A^{-1} = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

<u>Check</u>: $[T]_\beta \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ ✓

$[T]_\beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ ✓

$[T]_\beta \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓