

Lecture 3: Subspaces (§1.3 of [FIS]) ①

Previously on Math 416...

A vector space over \mathbb{R} is a set V with two operations (vector addition and scalar mult) satisfying: (1-2) vec. addition is commutative and associative.

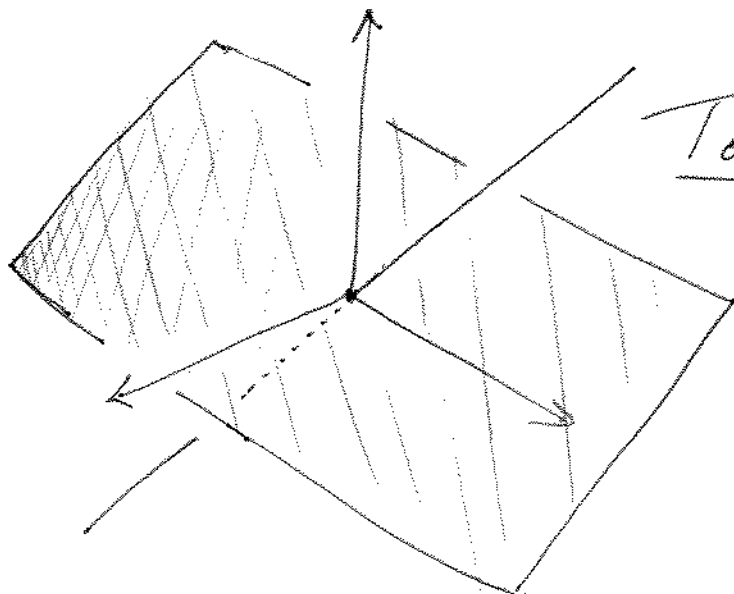
(3) There is a zero vector. (4) Additive inverses exist.

(5) $1v = v$ (6) scalar mult is assoc.

(7-8) Distributive properties.

Ex: \mathbb{R}^n , $\text{Mat}_{m \times n}(\mathbb{R})$, spaces of functions...

Back to \mathbb{R}^3 : Other basic objects: lines and planes.



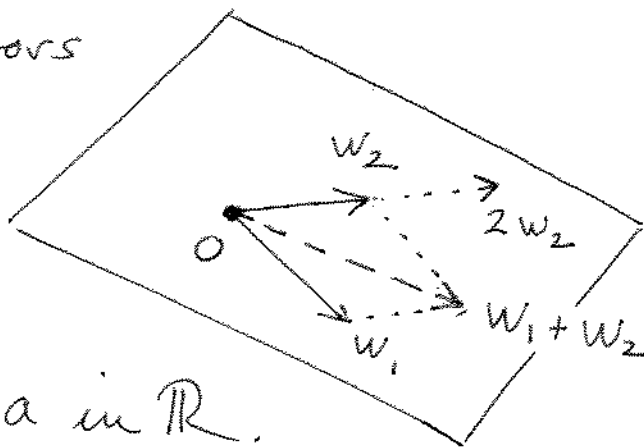
Today: Analog of such in a general vector space.

Suppose W is a plane in \mathbb{R}^3 containing 0 , (2)

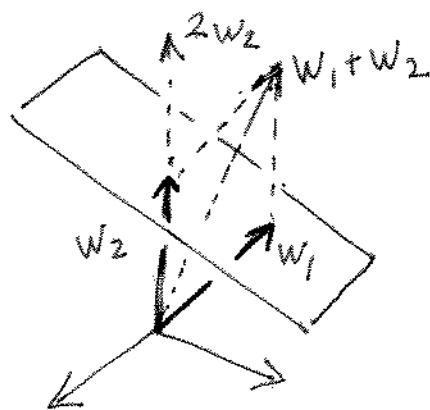
and w_1, w_2 are vectors in W . Then

$w_1 + w_2$ is also in W .

So is aw_1 , for any a in \mathbb{R} .



Note: Important that W contains 0 here as otherwise these props need not hold.



Def: Suppose V is a vector space over \mathbb{R} . A subset W of V is a

subspace if (a) 0 is in W

(b) For all w_1, w_2 in W , the sum $w_1 + w_2$ is also in W .

(c) For all c in \mathbb{R} and w in W , cw is also in W .

[Can replace (a) with requirement that W is nonempty.]

Ex: Some subspaces of \mathbb{R}^3 :

(3)

① \mathbb{R}^3 ② $\{0\}$ ③ $\{(x, 0, 0) \text{ for } x \text{ in } \mathbb{R}\}$

④ $\{(x, -x, 2x)\}$ ⑤ $\{(x, y, 0)\}$

⑥ $\{x+y+z=0\} = \{s(1, 0, -1) + t(1, -1, 0) \text{ for } s, t \text{ in } \mathbb{R}\}$

Ex: In any vector space V , the subsets $\{0\}$ and V are subspaces.

Thm: Suppose W is a subspace of a vector space V . Then W is itself a vector space under the two operations inherited from V .

Proof: First by requirements (a) and (b)

we do have two ops taking values in W .

Of the 8 conditions, (1-2) and (5-8)

are immediate from the fact that V

itself is a vector space. Moreover,

(3) follows from subspace cond. (a).

④

Finally, for ④ given w in W we know

there is a v in V such that $v+w=0$.

Issue: Does v have to be in W ?

Yes, since we can take $w=(-1)v$ which is in

W by ③. Check: $v+(-1)v \stackrel{⑤}{=} 1v+(-1)v \stackrel{⑧}{=} (1-1)v$

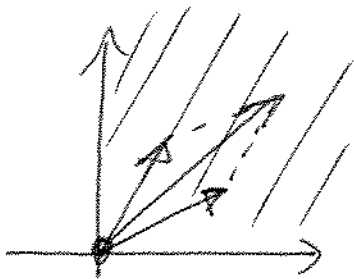
$$=0v=0$$

↑ Thm of last time.

So W with these ops satisfies ①-⑧ and

so is a vector space. ▣

Non-Ex: $W = \{(w_1, w_2) \text{ with } w_i \geq 0\}$ Proof end symbol
in \mathbb{R}^2 is not a subspace. (= Q.E.D.)



Satisfies ① and ② but not ③.

In proof of thm, everything works except ④.

[Discuss difference with book's treatment of subspaces.]

Ex: A in $\text{Mat}_{n \times n}(\mathbb{R})$

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & \\ \vdots & & \ddots & \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix} = (A_{ij})$$

Transpose: A^t where $A^t_{ij} = A_{ji}$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

[Also works for non-square matrices.]

A matrix A in $\text{Mat}_{n \times n}(\mathbb{R})$ is symmetric if

$$A = A^t.$$

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ but neither of the two examples above.

Thm: The subset of symmetric matrices in $\text{Mat}_{n \times n}(\mathbb{R})$ is a subspace.

⑥
Proof: The O in $\text{Mat}_{n \times n}(\mathbb{R})$ is $\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$
which is symmetric so (a) holds.

For (b) and (c), first show that for all A, B in $\text{Mat}_{n \times n}(\mathbb{R})$ and a, b in \mathbb{R} one has

$$(aA + bB)^t = a(A^t) + b(B^t).$$

Now if A, B are sym, then

$$(A+B)^t = A^t + B^t = A + B$$

and so $A+B$ is also sym, proving (b).

The argument for (c) is similar. \square

Next time: Linear combinations
and linear equations