Lecture 15: Composition of linear transformations and matrix multiplication

[§2.3 of FIS]

Last time:

$T: V \to W$ linear transformation

$\beta = \{v_1, \ldots, v_n\}$ a basis for V

$\gamma = \{w_1, \ldots, w_n\}$ a basis for W

$$[T]_\beta^\gamma = \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix} \in M_{m \times n}(\mathbb{R})$$

Notes: • For $T: V \to V$, we'll write $[T]_\beta$ for $[T]_\beta^\beta$.

• For vector spaces V and W, set

$$\mathcal{L}(V, W) = \{T: V \to W \mid T \text{ linear}\}$$

and

$$\mathcal{L}(V) = \mathcal{L}(V, V).$$

Fun Fact: $\mathcal{L}(V, W)$ is itself a vector space, ~~with~~ specifically a subspace of $\mathcal{F}(V, W)$. Come back to later, but for now details are in the text.

Consider two linear transformations:

$$X \xrightarrow{S} Y \xrightarrow{T} Z$$

<u>Thm</u>: Their composition $T \circ S : X \to Z$, defined by $(T \circ S)(x) = T(S(x))$, is also a linear trans.

<u>Pf</u>: Suppose $x_1, x_2 \in X$, $a \in \mathbb{R}$. Then

$$(T \circ S)(a x_1 + x_2) = T(S(a x_1 + x_2))$$

$$= T(a S(x_1) + S(x_2)) = a T(S(x_1)) + T(S(x_2))$$

$$= a (T \circ S)(x_1) + (T \circ S)(x_2).$$

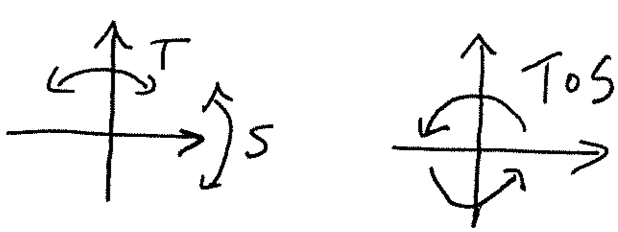
$\square$

<u>Ex</u>: $S, T : V \to V$ where $V = \mathbb{R}^2$

$S(x, y) = (-x, y)$ reflection in y-axis

$T(x, y) = (x, -y)$ reflection in x-axis

$(T \circ S)(x, y) = (-x, -y)$ rotation by angle π.



$\alpha = \{x_1, \dots, x_n\}$ basis for X

$\beta = \{y_1, \dots, y_m\}$ basis for Y

$\gamma = \{z_1, \dots, z_p\}$ basis for Z

Q: How is $[T \circ S]_\alpha^\gamma$ related to $[S]_\alpha^\beta$ and $[T]_\beta^\gamma$?

[Answer involves matrix multiplication...]

Def: Let A be a $p \times m$ matrix and B be an $m \times n$ matrix.

Their product AB is the $p \times n$ matrix defined by

$$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$$

Ex:

$$\underset{2 \times 3}{\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \end{pmatrix}} \underset{3 \times 2}{\begin{pmatrix} 1 & -3 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}} = \underset{2 \times 2}{\begin{pmatrix} 5 & -1 \\ 8 & 7 \end{pmatrix}}$$

If we view a column vector as a $m \times 1$ matrix, this agrees with the product of (matrix)(vector) defined last time.

Moreover

$$AB = A\begin{pmatrix} | & & | \\ b_1 & \cdots & b_n \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ Ab_1 & \cdots & Ab_n \\ | & & | \end{pmatrix}$$

<u>Thm</u>: $$\underset{p \times n}{[T \circ S]_\alpha^\gamma} = \underset{p \times m}{[T]_\beta^\gamma}\ \underset{m \times n}{[S]_\alpha^\beta}$$

<u>Pf</u>: Recall from last time that if $y \in Y$ then

$$[T(y)]_\gamma = [T]_\beta^\gamma [y]_\beta$$

Now

$$[S]_\alpha^\beta = \begin{pmatrix} | & & | \\ [S(x_1)]_\beta & \cdots & [S(x_n)]_\beta \\ | & & | \end{pmatrix}$$

so

$$[T]_\beta^\gamma [S]_\alpha^\beta = \underbrace{\begin{pmatrix} | \\ [T]_\beta^\gamma [S(x_i)]_\beta \\ | \end{pmatrix}}_{i=1,\ldots,n}$$

$$= \underbrace{\begin{pmatrix} | \\ [T(S(x_i))]_\gamma \\ | \end{pmatrix}}_{i=1,\ldots,n} = \underbrace{\begin{pmatrix} | \\ [(T \circ S)(x_i)]_\gamma \\ | \end{pmatrix}}_{i=1,\ldots,n}$$

$$= [T \circ S]_\alpha^\gamma \qquad \blacksquare$$

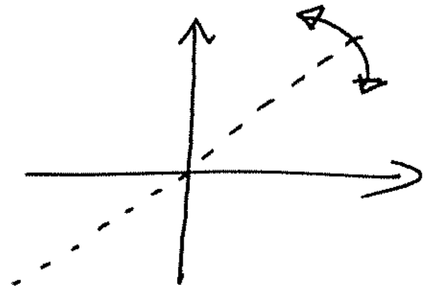
Ex: $S, T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad \beta = \{e_1, e_2\}$

$$T(x,y) = (x,-y) \qquad [T]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S(x,y) = (y,x) \qquad [S]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

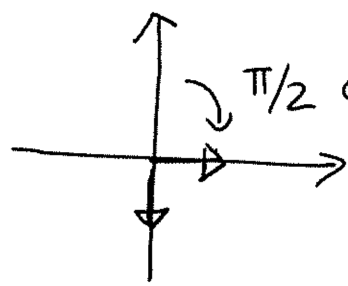
T is reflection in x-axis

S is reflection in $y = x$.



$$(T \circ S)(x,y) = T(S(x,y)) = T(y,x) = (y,-x)$$

$$[T \circ S]_\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$



$$[T]_\beta [S]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \checkmark$$

Important: Order of composition maters!

$$[S \circ T]_\beta = [S]_\beta [T]_\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

So $S \circ T$ is rotation by $\pi/2$ counter-clockwise.

<u>Thm</u>: Matrix multiplication is

<u>Associative</u>: $A(BC) = (AB)C$

<u>Distributative</u>: $A(B+C) = AB + AC$

$$(A+B)C = AC + BC$$

but not always commutative, as we just saw.

[Can prove this mechanically using formula for matrix mult. Will discuss a conceptual proof next time, but can also see the text.]

Suppose A is an $m \times n$ matrix. Then

$$A \cdot \underset{n \times p}{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}} = \underset{m \times p}{\begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}}$$

$$\underset{p \times m}{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}} A = \underset{p \times n}{\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}}$$

What about 1?

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Will start here next time...

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$