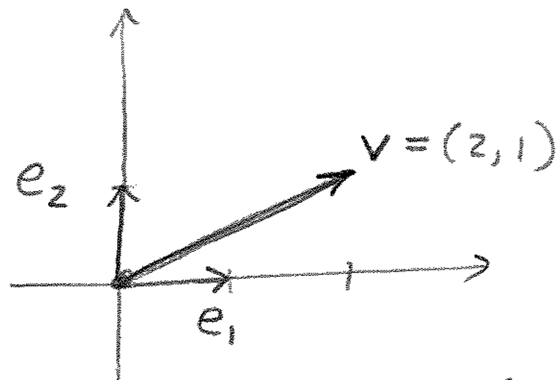
# Lecture 14: Encoding Linear Transformations as Matrices. [§2.2 FIS]

Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space $V$. For any $v \in V$ there are unique scalars such that

$$v = a_1 v_1 + \cdots + a_n v_n$$

The coordinate vector of $v$ with respect to $\beta$ is

$$[v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$




Ex: $V = \mathbb{R}^2$

$$\beta = \{e_1, e_2\} \qquad v = 2e_1 + 1 \cdot e_2 \implies [v]_\beta = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\beta' = \{e_2, e_1\} \qquad v = 1 \cdot e_2 + 2e_1 \implies [v]_{\beta'} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\gamma = \{u_1 = (1,1), u_2 = (1,-1)\}$$

$$v = \frac{3}{2}u_1 + \frac{1}{2}u_2 \implies [v]_\gamma = \begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix}$$

Note that $[\;]_\beta$ gives a map

$$V \longrightarrow \mathbb{R}^n$$
$$v \longmapsto [v]_\beta$$

which is

(a) Linear: $[a v_1 + v_2]_\beta = a[v_1]_\beta + [v_2]_\beta$

(b) a 1-1 and onto map, also called a bijective correspondence. The inverse map is $\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \longrightarrow a_1 v_1 + \cdots + a_n v_n$.

<u>Moral</u>: For a finite dim'l vecter space, a choice of basis gives a way to relate it to $\mathbb{R}^n$.

---

Now suppose $T: V \longrightarrow W$ is linear, and $\beta = \{v_1, \ldots, v_n\}$ is a basis for $V$

$\gamma = \{w_1, \ldots, w_n\}$ is a basis for $W$

From last time, ~~know~~ $\{T(v_1), \dots, T(v_n)\}$ determine $T$.

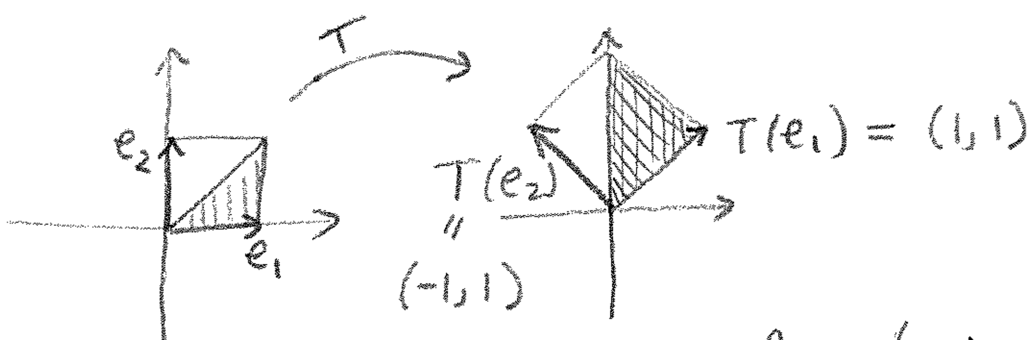
The <u>matrix</u> of $T$ <u>in the bases</u> $\beta$ <u>and</u> $\gamma$ is

$$[T]_\beta^\gamma = \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix} \in M_{m\times n}(\mathbb{R})$$

Equivalently, its entries are the unique scalars $a_{ij}$ where

$$T(v_j) = a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_j$$

<u>Ex</u>: $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad T(x,y) = (x-y, x+y)$




$$[T]_\beta^\beta = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad [T]_{\beta'}^\beta = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$[T]_\beta^{\beta'} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad [T]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$T(e_1) = e_2 + e_1$ $\qquad$ $T(e_1) = u_1$

$T(e_2) = e_2 - e_1$ $\qquad$ $T(e_2) = -u_2$

$$[T]_\gamma^\gamma = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$T(u_1) = (0,2) = u_1 - u_2$

$T(u_2) = (2,0) = u_1 + u_2$

---

Suppose $A \in M_{m\times n}$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ a column vector of height $n$. Their product $Ax$ is the column $y$ vector of height $m$ where

$$y_i = A_{i1}x_1 + A_{i2}x_2 + \cdots + A_{in}x_n = \sum_{j=1}^{n} A_{ij}x_j$$

Ex: $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -3 & 4 \end{pmatrix}$ $\quad x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad Ax = \begin{pmatrix} 2\cdot 1 + 1\cdot 2 + 0\cdot 3 \\ 0\cdot 1 - 3\cdot 2 + 4\cdot 3 \end{pmatrix}$

$$= \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

Ex: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 11 \end{pmatrix}$ $\qquad \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Relevance: ① $\mathcal{LS}(A, b)$ is $A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = b$

↑ variables

② $T: V \to W$ with $\beta$ a basis for $V$
$\gamma$ a basis for $W$

Then

$$[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$$

[Will justify momentarily, but first let's use.]

<u>Ex</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad T(x,y) = (x-y,\ x+y)$

$\beta = \{e_1, e_2\}$

$v = (2,1)$ $\quad \gamma = \{u_1 = (1,1),\ u_2 = (1,-1)\}$

- $[T(v)]_\beta = [(1,3)]_\beta = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

  $[T]_\beta^\beta [v]_\beta = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ ✓

- $[T(v)]_\gamma = [(1,3)]_\gamma = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

  $[T]_\gamma^\gamma [v]_\gamma = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 3/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ✓

- $[T]_\beta^\gamma [v]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ✓

<u>Proof</u> that $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$

$\beta = \{v_1, \dots, v_n\}$ $\gamma = \{w_1, w_2, \dots, w_m\}$

If $v = a_1 v_1 + \cdots + a_n v_n$ then by linearity have

$$T(v) = a_1 T(v_1) + \cdots + a_n T(v_n).$$

Now the map $\begin{aligned} W &\longrightarrow \mathbb{R}^m \\ w &\longmapsto [w]_\gamma \end{aligned}$ is itself linear,

so

$$\begin{aligned} [T(v)]_\gamma &= a_1 [T(v_1)]_\gamma + \cdots + a_n [T(v_n)]_\gamma \\ &= \begin{pmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ &= [T]_\beta^\gamma [v]_\beta \end{aligned}$$

$\blacksquare$

<u>Thm</u>: Suppose $V$ and $W$ are vector spaces with basis $\beta, \gamma$ with $n = \#\beta$ and $m = \#\gamma$. Given $A \in M_{m \times n}(\mathbb{R})$, there is a unique linear transformation $T: V \longrightarrow W$ with $[T]_\beta^\gamma = A$.

<u>Pf</u>: Later this week.