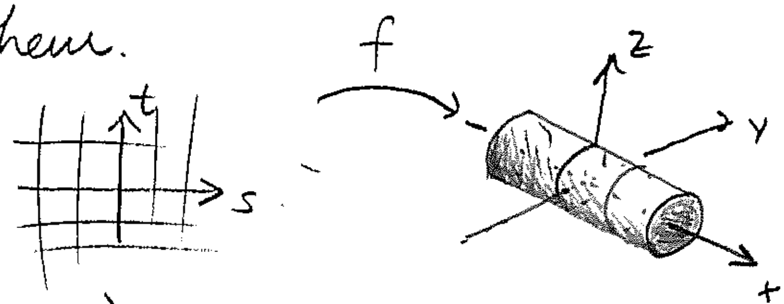
Lecture 12: Linear Transformations, [FIS Ch 2]
an introduction

Now that we've studied vector spaces, we look at functions between them.



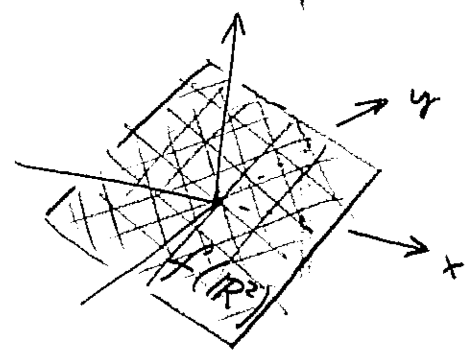
Ex: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f(s,t) = (s, \cos t, \sin t)$$

[In this class, we'll focus on the simplest kind of functions between vector spaces, for example.]

Ex: $f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$f(s,t) = (s, t, s+t)$$



Def: Suppose V and W are vector spaces over $\mathbb{R}$. A function $T: V \longrightarrow W$ is a linear transformation if for all $v_1, v_2 \in V$ and $a \in \mathbb{R}$ we have

a) $T(v_1 + v_2) = T(v_1) + T(v_2)$.

b) $T(av_1) = aT(v_1)$.

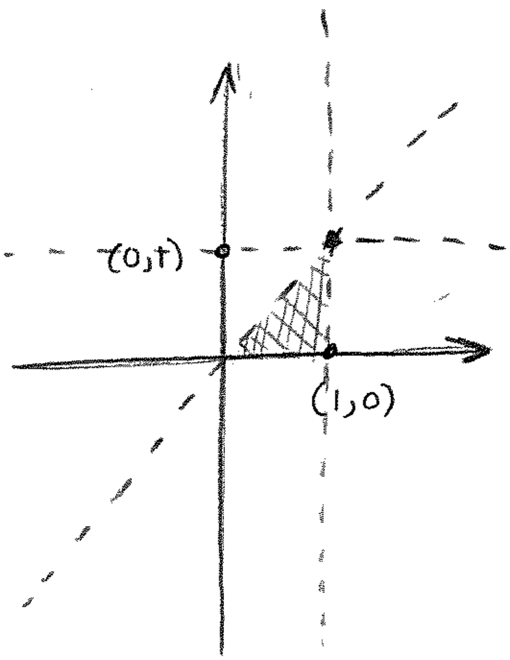
Ex: $T : \mathbb{R}^2 \to \mathbb{R}^2$

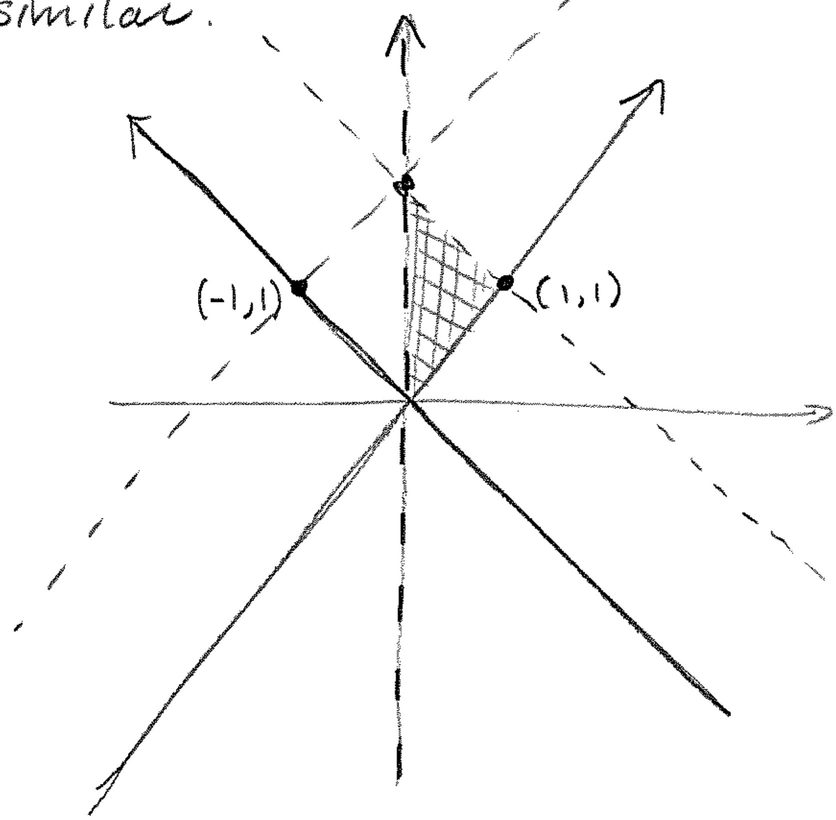
$T(x,y) = (x-y,\ x+y)$ Let's check this is a linear trans.

$v_1 = (x_1, y_1)$ $v_2 = (x_2, y_2)$

$$
\begin{aligned}
T(v_1+v_2) &= T((x_1+x_2,\ y_1+y_2)) \\
&= ((x_1+x_2)-(y_1+y_2),\ (x_1+x_2)+(y_1+y_2)) \\
&= (x_1-y_1,\ x_1+y_1) + (x_2-y_2,\ x_2+y_2) \\
&= T(v_1) + T(v_2).
\end{aligned}
$$

Scalar mult is similar.





$T(x,0) = (x,x)$

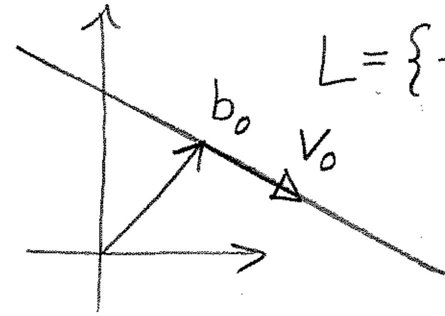
$T(0,y) = (-y,y)$

$T(t,t) = (0,2t)$

$T(1,1) = 2$

By prop (b), namely $T(av) = aT(v)$ see that T must send a line through 0 to another one. In fact, this is true for *any* line in $\mathbb{R}^2$:

$L = \{ tv_0 + b_0 \mid t \in \mathbb{R} \}$



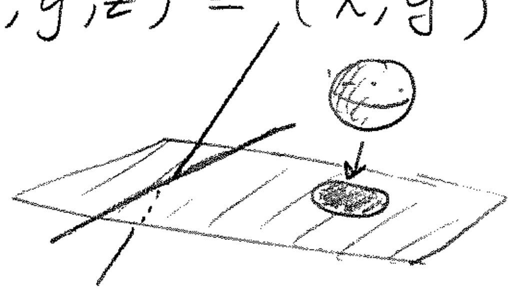
$$T(tv_0 + b_0) =$$

$$T(tv_0) + T(b_0) = t\,T(v_0) + T(b_0)$$

[Draw some examples]

Ex: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad T(x,y) = (x, y, x+y)$ [From before]

Ex: $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\quad T(x,y,z) = (x,y)$



Ex: $T: P(\mathbb{R}) \to P(\mathbb{R})$

$$T(f(x)) = f'(x)$$

$$T(x^3 + 2x^2 + 5x + 1) = 3x^2 + 4x + 5$$

[The fact that this is a linear transformation is one of the first things you learned about the derivative in Calc I]

Ex: $T : M_{m\times n}(\mathbb{R}) \longrightarrow M_{n\times m}(\mathbb{R})$

$$A \xmapsto{\text{"sends"}} A^t$$

[Used implicitly to show symmetric matrices are a subspace of $M_{n\times n}(\mathbb{R})$.]

Def: Suppose $T : V \to W$ is a linear transformation.

The nullspace (kernel) of T is

$$\mathcal{N}(T) = \{v \in V \mid T(v) = 0_W\}.$$

The range (image) of T is

$$R(T) = \{T(v) \mid v \in V\}$$

Thm. These are subspaces of V and W respectively.

Pf: For $\mathcal{N}(T)$:

i) $T(0_V) = T(0 \cdot 0_V) = 0 \cdot T(0_V) = 0_W$, so $0_V \in \mathcal{N}(T)$.

ii) Suppose v_1, v_2 are in $\mathcal{N}(T)$. Then

$$T(v_1 + v_2) = T(v_1) + T(v_2) = 0_W + 0_W = 0_W$$

and so $v_1 \in \mathcal{N}(T)$.

iii) Suppose $a \in \mathbb{R}$ and $v_1 \in \mathcal{N}(T)$. Then

$$T(av_1) = aT(v_1) = a \cdot 0_W = 0_W$$

and so $av_1 \in \mathcal{N}(T)$.

The case of $R(T)$ is similar. ▨

[Go back to examples, figure out nullsp, range.]

<u>Dimension Theorem</u>: Suppose $T: V \to W$ is a linear transformation. If V is finite dim'l, then

$$\underbrace{\dim(\mathcal{N}(T))}_{\text{nullity}} + \underbrace{\dim(R(T))}_{\text{rank}} = \dim V.$$

[Again revisit the examples.]

Encoding $T: V \longrightarrow W$ a linear transformation

Suppose $\beta = \{v_1, \dots, v_n\}$ is a basis for V.

Knowing $T(v_i)$ for $i = 1, \dots, n$ determines T since by repeatedly applying the linearity props we get

$$T(v = b_1 v_1 + \cdots + b_n v_n) = b_1 T(v_1) + \cdots + b_n T(v_n)$$

If $\gamma = \{w_1, \dots, w_m\}$ is a basis for W, we can write

$$T(v_j) = a_{1j} w_1 + a_{2j} w_2 + \cdots + a_{mj} w_j$$

Hence we can encode T completely by the matrix

$$[T]_\beta^\gamma = (a_{ij}) \in M_{m \times n}(\mathbb{R}).$$

Ex: $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$T(x,y) = (x+2y,\ 3x+4y)$$

$$\beta = \gamma = \{ e_1 = (1,0),\ \ e_2 = (0,1) \}$$

$$T(e_1) = (1,3) \qquad T(e_2) = (2,4)$$

$$= e_1 + 3e_2 \qquad\qquad = 2e_1 + 4e_2$$

$$[T]_\beta^\beta = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

[If time remains, go back and do the earlier examples.]