Lecture 11: Basis, dimension, and linear systems.

Story so far:

Vector spaces/subspaces:

linear combinations

$\rightsquigarrow$ span, linear (in)dependence

$\rightsquigarrow$ basis, dimension

Linear Systems:

Matrix encoding.

Row ops, RREF, parametrizing solutions.

Connections:

Write v as a linear comb of $u_1, u_2, \dots, u_k$

$$\rightsquigarrow \mathcal{LS}\left(\left(\begin{array}{ccc|c} | & & | & | \\ u_1 & \cdots & u_k & v \\ | & & | & | \end{array}\right)\right)$$

$\{u_1, \dots, u_k\}$ linearly dependent

$$\Longleftrightarrow \text{Nullspace of } \begin{pmatrix} | & & | \\ u_1 & \cdots & u_k \\ | & & | \end{pmatrix}$$

contains a non-zero vector.

Today:
- Basis and dim of $\mathcal{N}(A)$.
- Find a basis for $\text{span}(\{u_1, \dots, u_k\})$.

[Second topic will be another application of reduced row echelon form.]

A an $m \times n$ matrix

HW2!

$\mathcal{N}(A)$ is the subspace of $\mathbb{R}^n$ consisting of solutions to $\underbrace{\mathcal{LS}(A, 0)}_{\text{a } \underline{\text{homogenous}} \text{ linear system}} = \mathcal{LS}\left(\left(A \,\middle\vert\, \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right)\right)$

<u>Ex</u>:

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 4 & -1 \\ -1 & -2 & 1 & -2 \end{pmatrix} \xrightarrow{\text{row ops}} \begin{array}{c} \begin{matrix} & s & & t \end{matrix} \\ \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$\mathcal{N}(A) = \{(-2s-t, s, t, t) \mid s, t \in \mathbb{R}\}$$

<u>Claim</u>: $\beta = \left\{ \begin{matrix} u_1 = (-2, 1, 0, 0) \\ u_2 = (-1, 0, 1, 1) \end{matrix} \right\}$ is a basis for $\mathcal{N}(A)$

<u>Check</u>: <u>Span</u>:

$$(-2s-t, s, t, t) = s u_1 + t u_2 \checkmark$$

<u>Linear independence</u>: If $a_1 u_1 + a_2 u_2 = 0$, get $a_1 = 0$ from the 2^{nd} coordinate and $a_2 = 0$ from 3^{rd} or 4^{th} coordinate.

<u>Thm</u>: Suppose $A \in Mat_{m \times n}(\mathbb{R})$ is in RREF with p pivot columns. Then

$\dim \mathcal{N}(A) = n - p = $ # of non-pivot columns.

<u>Proof</u>: The linear system is consistrent with $n-p$ free variables. Let

$u_k = $ Solution where k^{th} free var is 1 and others are 0.

<u>Claim</u>: $\{u_1, \ldots, u_{n-p}\}$ is a basis for $\mathcal{N}(A)$.

This spans, as for any scalars $t_1, \ldots, t_{n-p}$, the vector

$$t_1 u_1 + \cdots + t_{n-p} u_{n-p}$$

is the unique solution to $\mathcal{LS}(A, 0)$ where the free vars have values $t_1, \ldots, t_{n-p}$.

They are linearly independent as only u_k has a non-zero entry in the position corresponding to the k^{th} free variable.

$\blacksquare$

Suppose $A \in \mathrm{Mat}_{m \times n}(\mathbb{R})$. The <u>row space</u> of A is the span of its rows in $\mathbb{R}^n$, and is denoted $\mathcal{R}(A)$.

<u>Ex</u>: $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 1 & 1 \end{pmatrix}$ $\quad \mathcal{R}(A) = \mathrm{span}(\{(2,1),(0,3),(1,1)\})$

$$= \mathbb{R}^2$$

[Query!]

<u>Thm</u>: If A and B are row equivalent matrices, then $\mathcal{R}(A) = \mathcal{R}(B)$.

<u>Pf</u>: Enough to show that doing a single row operation on a matrix M does not change span(M). Interesting case: $M = \begin{pmatrix} \text{---} & r_1 & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{pmatrix}$ and replace r_1 with $r_1 + c r_2 = u$ to get $N = \begin{pmatrix} \text{---} & u & \text{---} \\ \text{---} & r_2 & \text{---} \\ & \vdots & \\ \text{---} & r_m & \text{---} \end{pmatrix}$.

As $u, r_2, \dots, r_m \in \mathcal{R}(M)$, have

$\mathcal{R}(N) \subseteq \mathcal{R}(M)$. As $r_1 = u - c r_2$

have $r_1, r_2, \ldots, r_m \in R(N)$ and so $R(M) \subseteq R(N)$.

Hence $R(N) = R(M)$, as desired. $\blacksquare$

<u>Thm</u>: If $A \in Mat_{m \times n}(\mathbb{R})$ is in RREF, then the <u>nonzero</u> rows of A are a basis for $R(A)$.

<u>Proof</u>: Let $r_1, \ldots, r_k$ be the nonzero rows of A. By definition, they span $R(A)$. To see they're linearly independent, note that r_i is the only row with a nonzero entry in the position j of its leading 1. $\blacksquare$

Ex: What is a basis for

$$\text{span}(\{(0,1,2),\ (3,4,5),\ (6,7,8),\ (9,10,11)\})$$

in $\mathbb{R}^3$?

Note: Compare with check for linear depend!

Set $A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \\ 9 & 10 & 11 \end{pmatrix}$. We seek a basis for $R(A)$.

Row reduce to $B = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

So

$$R(A) = R(B)$$

has basis $\beta = \{(1,0,-1),\ (0,1,2)\}$