# Lecture 28:

Last time: $\lambda$ an eigenvalue of $A \in M_{n \times n}$.

Algebraic Mult: # of times $(t-\lambda)$ divides char poly of $A$.

Geometric Mult: $\dim(E_\lambda)$.

Thm: A matrix $A \in M_{n\times n}(\mathbb{R})$ is diagonalizable if and only if a) The char poly of $A$ splits completely over $\mathbb{R}$.
b) For all eigenvalues of $A$, (alg mult) = (geom mult).

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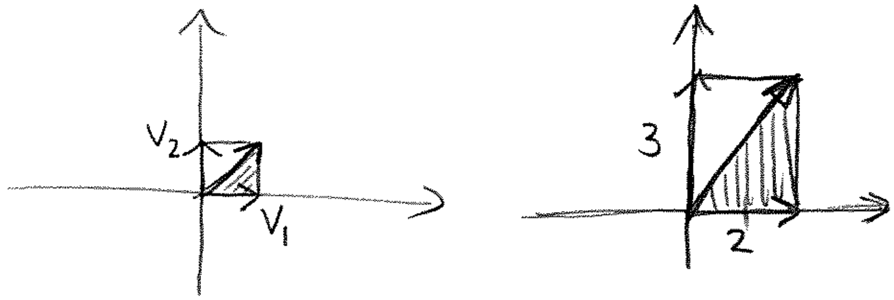
Lemma: Suppose $v_1, \dots, v_k \in \mathbb{R}^n$ are eigenvectors of $A$ corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then $\{v_1, \dots, v_k\}$ is linearly independent.

Moral: Can't create an eigenvector with with eigenvalue $\lambda$ from eigenvectors with other eigenvalues.

Ex: $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$




Proof of Lemma: Induct on $k$.

Base case: As $v_1$ is an eigenvector, $v_1 \neq 0$ and so $\{v_1\}$ is linearly independent.

Inductive Step: Assume $\{v_1, \ldots, v_{k-1}\}$ is linearly independent. Suppose there are scalars with

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0 \qquad (1)$$

Multiplying both sides by $A$ gives

$$a_1 \lambda_1 v_1 + \cdots + a_k \lambda_k v_k = A0 = 0 \qquad (2)$$

Considering $-\lambda_k (1) + (2)$ gives

$$a_1(\lambda_1 - \lambda_k) v_1 + \cdots + a_{k-1}(\lambda_{k-1} - \lambda_k) v_k = 0$$

So for $i < k$ have $a_i(\lambda_i - \lambda_k) = 0$;

as $\lambda_i \neq \lambda_k$ this forces $a_i = 0$ for $i < k$.

Thus (1) gives $a_k v_k = 0$ which implies $a_k = 0$

So all $a_i = 0$ and $\{v_1, \dots, v_k\}$ is linearly independent, completing the induction. ▨

<u>Proof of Thm</u>: ($\Longrightarrow$) By last time, know

$$\text{char poly } A = (\lambda_1 - t)^{m_1} \cdots (\lambda_k - t)^{m_k}$$

where the $m_i$ are the distinct eigenvalues of $A$ and $\sum m_i = n$. Set $d_i = \dim E_{\lambda_i}$.

Must show each $d_i = m_i$ and already know that $d_i \le m_i$. Let $\beta$ be a basis of $\mathbb{R}^n$ consisting of eigenvectors for $A$. Set $c_i = \#\{v \in \beta \mid v \in E_{\lambda_i}\}$. As as any subset of $\beta$ is linearly indep, must have

$$c_i \le d_i$$

Now

$$n = \sum c_i \le \sum d_i \le \sum m_i = n$$

which forces $d_i = m_i$ for all $i$ as req'd.

($\Leftarrow$) Let $\lambda_i$, $d_i$, $m_i$ be as above. As the char poly splits completely, have $\sum m_i = n$, and by assumption $m_i = d_i$ for each $i$.

Let $\beta_i$ be a basis for $E_{\lambda_i}$.

<u>Claim</u>: $\beta = \beta_1 \cup \cdots \cup \beta_K$ is a basis for $\mathbb{R}^n$.

If so, then $A$ is diagonalizable as desired.

Now $E_{\lambda_i} \cap E_{\lambda_j} = \{0\}$ if $i \neq j$, so

$$\#\beta = \sum_{i=1}^{k} \#\beta_i = \sum_{i=1}^{k} m_i = n$$

and thus it suffices to show that $\beta$ is linearly independent. Suppose

$$\beta_i = \{v_1^i, v_2^i, \ldots, v_{m_i}^i\}$$

and there are scalars $a_j^i$ where

$$\sum_{i=1}^{k} \underbrace{\left( \sum_{j=1}^{m_i} a_j^i v_j^i \right)}_{w_i} = 0$$

Each $w_i$ is either 0 or an eigenvector corr. to $\lambda_i$. By lemma, can't have a linear dependence among eigenvectors with different eigenvalues, so must have all $w_i = 0$. As each $\beta_i$ is linearly independent, must have $a_1^i, \ldots, a_{m_i}^i$ all 0.

So all $a_j^i = 0$ and so $\beta$ is linearly independent. This proves the claim and thus the theorem. Q.E.D.