

Lecture 33: Gram-Schmidt and friends §6.2 (1)

Last time: V an inner product space.

- $S \subseteq V$ is orthogonal if $\langle x, y \rangle = 0$ for all distinct $x, y \in S$.
- If additionally all x in S are unit, then S is called orthonormal.

Thm: Suppose $S = \{v_1, v_2, \dots, v_k\} \subseteq V$ is orthonormal.

If $y \in \text{span}(S)$ then $y = \sum \langle y, v_i \rangle v_i$. Moreover, S is linearly independent.

Thm: Suppose V is a finite dim'l inner product space. Then V has an orthonormal basis.

[Prove today via the algorithm that builds such a basis...]

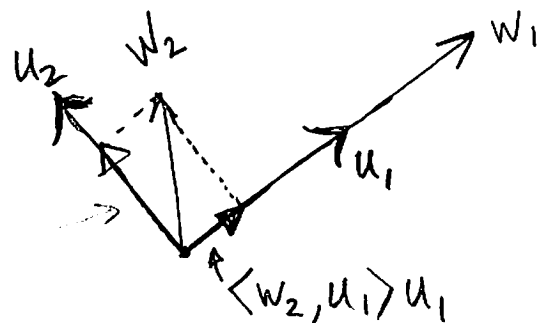
Gram-Schmidt Process: $\{w_1, \dots, w_n\}$ linearly indep.

$$u_1 = \text{unit}(w_1) = \frac{w_1}{\|w_1\|}$$

$$u_2 = \text{unit} \left(w_2 - \langle w_2, u_1 \rangle u_1 \right)$$

⋮

$$u_k = \text{unit} \left(w_k - \sum_{i=1}^{k-1} \langle w_k, u_i \rangle u_i \right)$$



Claim: $\{u_1, \dots, u_n\}$ is orthonormal with the same span as $\{w_1, \dots, w_n\}$

Note: By taking $\{w_1, \dots, w_n\}$ to be a basis of V then proves the theorem.

Pf of Claim: Induct on n .

$n=1$: As $\text{span}(\{u_1\}) = \text{span}(\{w_1\})$ and any single unit vector is orthonormal the claim holds.

$n=2$: Set $v_2 = w_2 - \langle w_2, u_1 \rangle u_1$ so that

$u_2 = \text{unit}(v_2) = \frac{v_2}{\|v_2\|}$. Now $\{u_1, u_2\}$ is orthonormal

as

$$\begin{aligned} \langle v_2, u_1 \rangle &= \langle w_2, u_1 \rangle - \langle \langle w_2, u_1 \rangle u_1, u_1 \rangle \\ &= \langle w_2, u_1 \rangle - \langle w_2, u_1 \rangle \underbrace{\langle u_1, u_1 \rangle}_{=1} = 0. \end{aligned}$$

As $\{u_1, u_2\}$ is linearly independent,

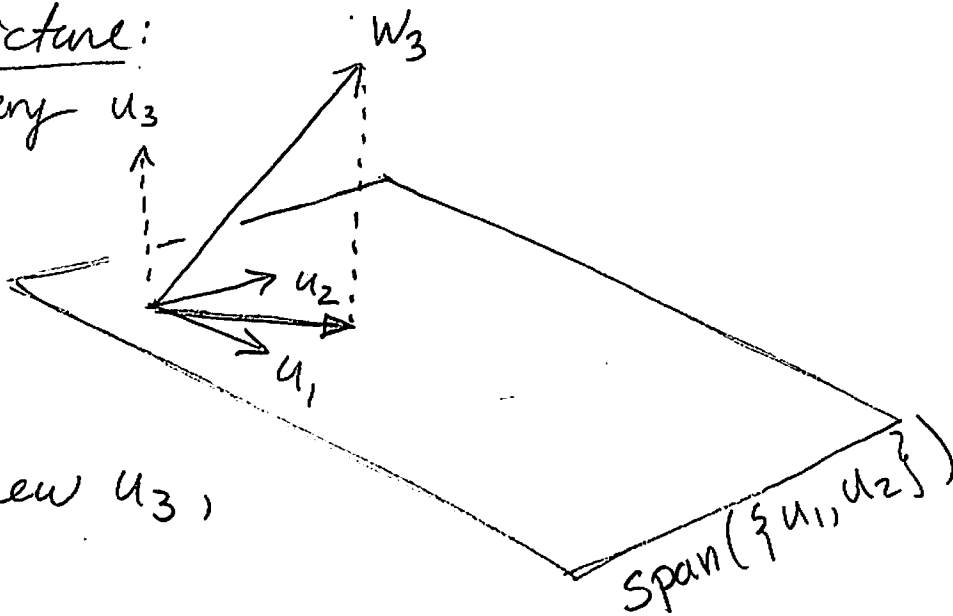
have $\dim(\text{span}\{u_1, u_2\}) = 2$. As $\text{span}\{u_1, u_2\} \subseteq \text{span}\{w_1, w_2\}$ this forces $\text{span}\{u_1, u_2\} = \text{span}\{w_1, w_2\}$ as needed.

The general inductive step is essentially the same as for $n=2$:

- Check that u_n is orthogonal to $\{u_1, \dots, u_{n-1}\}$.
- Note $u_n \in \text{span}(\{u_1, \dots, u_{n-1}, w_n\}) = \text{span}(\{w_1, \dots, w_n\})$ by induction and as $\{u_1, \dots, u_n\}$ is linearly indep we must have $\text{span}(\{u_1, \dots, u_n\}) = \text{span}(\{w_1, \dots, w_n\})$ for dimension reasons. ▣

Geometric Picture:

Mystery



If we knew u_3 ,

then

$$w_3 = \underbrace{\langle w_3, u_1 \rangle u_1 + \langle w_3, u_2 \rangle u_2}_{\text{Can calculate without knowing what } u_3 \text{ is.}} + \langle w_3, u_3 \rangle u_3$$

Can calculate without knowing what u_3 is.

Now solve for u_3 !
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(4)

Ex: $V = P_2(\mathbb{R})$ $\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$

$$\{1, x, x^2\} \xrightarrow[\text{Schmidt}]{\text{Gram}} \left\{ \frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2-1) \right\}$$

$$\langle 1, 1 \rangle = \int_{-1}^1 1 \cdot 1 dx = 2 \Rightarrow \|1\| = \sqrt{2}$$

These are called Legendre polynomials, the next few are $\sqrt{\frac{7}{8}}(5x^3 - 3x)$, $\sqrt{\frac{9}{128}}(35x^4 - 30x^2 + 3)$, ...

[First appeared in study of series expansions of gravitational potential functions in spherical coord.]

Def: Suppose $S \subseteq V$ is orthonormal. For

$x \in V$, the scalars $\langle x, u \rangle$ with $u \in S$ are called the Fourier coefficients of x relative

to S . continuous, real valued fns.

Reason for terminology: $V = \mathcal{C}([-1, 1])$

and $\langle f(x), g(x) \rangle = \int_{-1}^1 f(x)g(x) dx$

Then $S = \left\{ \frac{1}{\sqrt{2}} \right\} \cup \left\{ \sin(\pi n x) \right\}_{n=1}^{\infty} \cup \left\{ \cos(\pi n x) \right\}_{n=1}^{\infty}$

is an orthonormal subset of V . Consider

(5)

$h(x) = |x|$ in V . The Fourier coefficients are

$$\langle h, \frac{1}{\sqrt{2}} \rangle = \int_{-1}^1 |x| \frac{1}{\sqrt{2}} dx = \sqrt{2} \int_0^1 x dx = \frac{1}{\sqrt{2}}$$

$$\langle h, \sin \pi n x \rangle = 0 \text{ for symmetry reasons}$$

$$\begin{aligned} \langle h, \cos \pi n x \rangle &= \int_{-1}^1 |x| \cos \pi n x dx = 2 \int_0^1 x \cos(\pi n x) dx \\ &= \begin{cases} 0 & n \text{ even} \\ -\frac{4}{n^2 \pi^2} & n \text{ odd} \end{cases} \end{aligned}$$

Note: S is not a basis of V and infinitely many of these Fourier coefficients are non-zero.

However, if we allow infinite sums, get

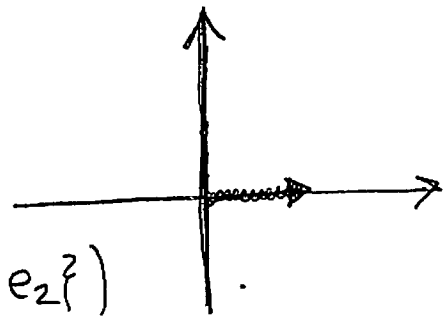
$$|x| = \frac{1}{2} + \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} -\frac{4}{n^2 \pi^2} \cos(\pi n x)$$

for all $x \in [-1, 1]$. Fun corollary: $\frac{\pi^2}{8} = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{n^2}$

Def: Suppose $S \subseteq V$ is nonempty. Then

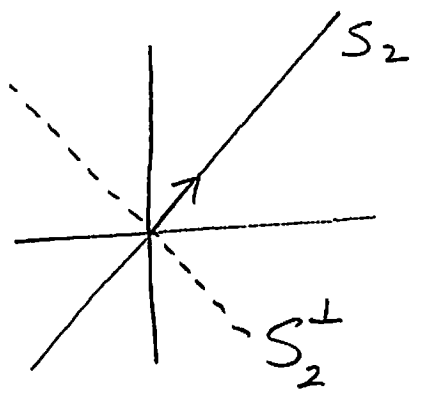
$S^\perp = \{x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in S\}$ is called the orthogonal complement of S .

Ex: $(\mathbb{R}^2, \text{dot prod})$



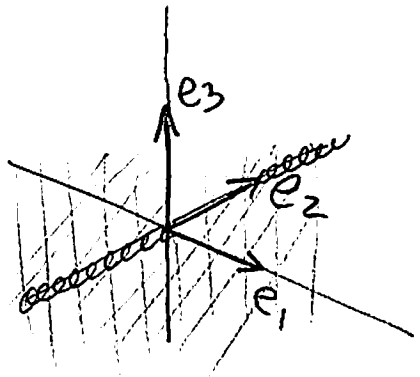
$S_1 = \{e_1\}, S_1^\perp = \text{span}(\{e_2\})$

$S_2 = \text{span}(\{(1, 1)\}), S_2^\perp = \text{span}(\{(-1, 1)\})$



Note: S^\perp is a subspace of V .

Ex: $(\mathbb{R}^3, \text{dot prod})$



$S_1 = \{e_3\}, S_1^\perp = \{xy \text{ plane}\}$

$S_2 = \{e_1, e_3\}, S_2^\perp = \{y\text{-axis}\}$