

# Lecture 27: Diagonalization Criteria [§5.2] ①

Thm:  $A \in M_{n \times n}(\mathbb{R})$  is diagonalizable if and only if there is a basis of  $\mathbb{R}^n$  consisting of eigenvectors for  $A$ .

Def: If  $\lambda$  is an eigenvalue for  $A \in M_{n \times n}(\mathbb{R})$

then  $E_\lambda = \{v \in \mathbb{R}^n \mid Av = \lambda v\} = \mathcal{N}(A - \lambda I_n)$   
is called the eigenspace of  $A$  corresponding to  $\lambda$ .

Today's Moral: There are two things that can prevent diagonalization

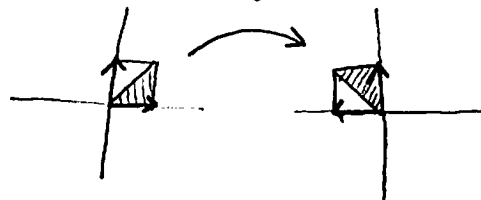
① Too few eigenvalues:  $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} \text{Char poly} &= \det(A - tI_2) = \det \begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} \\ &= t^2 + 1. \end{aligned}$$

This has no roots in  $\mathbb{R}$ , so  $A$  has no eigenvalues/eigenvectors at all!

(2)

Note: Geometrically,  $L_A: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is rotation by  $\pi/2$ , so clearly no eigenvectors.



Potential fix: Enlarge field to  $\mathbb{C} = \{a+bi \mid a, b \in \mathbb{R}\}$  where  $i^2 = -1$ . See HW for more.

② Too few eigenvectors:  $B = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ . Then char poly =  $(t-1)^2$  and so all eigenvectors live in:

$$E_1 = \mathcal{N}(B - I_2 = \begin{pmatrix} 0 & \\ & 0 \end{pmatrix}) = \{(t, 0) \mid t \in \mathbb{R}\}$$

So any eigenvector is a scalar mult of any other  $\Rightarrow$  Can't have a basis of eigenvectors, so not diagonalizable.

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Def: A polynomial  $f(t)$  with coefficients in a field  $\mathbb{F}$  splits (or splits completely) over  $\mathbb{F}$  if

$$f(t) = c(t - a_1)(t - a_2) \cdots (t - a_d)$$

where  $c, a_1, \dots, a_d \in \mathbb{F}$  and  $d = \deg(f)$ .

Fundamental Theorem of Algebra: Every polynomial with coeffs in  $\mathbb{C}$  splits completely over  $\mathbb{C}$ .

Thm: If  $A \in M_{n \times n}(\mathbb{R})$  is diagonalizable, then its char. poly splits completely over  $\mathbb{R}$ .

Proof: As  $A$  is similar to a diagonal matrix,

there is a  $Q \in M_{n \times n}(\mathbb{R})$  with  $Q^{-1}AQ = D$

$$= \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}. \quad \text{Then } \det(A - tI_n) =$$

$$\det(QDQ^{-1} - Q(tI_n)Q^{-1}) =$$

$$\det(Q(D - tI_n)) = \det \begin{pmatrix} \lambda_1 - t & & & 0 \\ & \lambda_2 - t & & \\ & & \ddots & \\ 0 & & & \lambda_n - t \end{pmatrix}$$

$$= (\lambda_1 - t)(\lambda_2 - t) \cdots (\lambda_n - t)$$

and so the char poly of  $A$  splits completely. ▣

[This gives a necessary condition for  $A$  to be diagonalizable. Is it also sufficient? No!]

Def: Suppose  $\lambda$  is an eigenvalue of  $A$ . The algebraic multiplicity of  $\lambda$  is the number of times that  $(t - \lambda)$  divides the char poly of  $A$ . The geometric multiplicity of  $\lambda$  is  $\dim(E_\lambda)$ .

Thm: For each  $\lambda$ ,  $(\text{geom mult}) \leq (\text{alg mult})$

Thm: A matrix  $A \in M_{n \times n}(\mathbb{R})$  is diagonalizable if and only if

a) char poly of  $A$  splits completely.

b) For every eigenvalue  $\lambda$ ,  $(\text{geom mult}) = (\text{alg mult.})$

(5)

Cor: If the char poly of  $A \in M_{n \times n}(\mathbb{R})$  has  $n$  distinct roots in  $\mathbb{R}$  then  $A$  is diagonalizable.

Pf of Cor: If the char poly has  $n$  distinct roots and degree  $n$ , then it splits completely.

Moreover every eigenvalue has alg. mult. 1.

As the geom mult of any  $\lambda$  is  $\geq 1$  we thus have that all alg and geom. mult. agree and so the theorem applies.  $\square$

Pf of Thm about multiplicities: Let

$\{v_1, \dots, v_k\}$  be a basis for  $E_\lambda$ . Enlarge this to a basis  $\beta = \{v_1, v_2, \dots, v_n\}$  for  $\mathbb{R}^n$ . Then

$$[L_A]_\beta = \underbrace{\left( \begin{array}{ccc|cc} \lambda & 0 & & & \\ & \lambda & & & \\ 0 & & \lambda & & \\ \hline & & & * & \\ & & & & * \end{array} \right)}_k \quad \text{as } L_A(v_i) = \lambda v_i \text{ for } 1 \leq i \leq k.$$

Setting  $Q = [I_{\mathbb{R}^n}]_\beta^{\text{std}} = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$ , have

⑥

$[L_A]_{\beta} = Q^{-1}AQ$ . As before,  
 we see similar matrices have the same char.  
 poly, and so

$$\begin{aligned}
 \text{char poly } A &= \det([L_A]_{\beta} - tI_n) \\
 &= \det \begin{pmatrix} \lambda-t & 0 & \vdots & * \\ 0 & \ddots & \lambda-t & \vdots \\ \vdots & & & \vdots \\ 0 & & & ? \end{pmatrix} \\
 &= (\lambda-t)^k \det(?)
 \end{aligned}$$

Thus (alg mult  $\lambda$ )  $\geq k$  as required. ▣