Lecture 41

Goal Thm: G finite gp. Then $\exists$ a Galois extension $K/\mathbb{C}(t)$ with group G.

Plan: ① Find a curve V in $\mathbb{P}^n_{\mathbb{C}}$ on which G acts by symmetries, so that $V/G = \mathbb{P}^1_{\mathbb{C}}$.

② G is now a subgp of $\mathrm{Aut}(K = \mathbb{C}(V))$.

③ $K_G = \mathbb{C}(V)_G = \mathbb{C}(V/G) = \mathbb{C}(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{C}(t)$.

Given a group G, let's make it act on some geometric object.

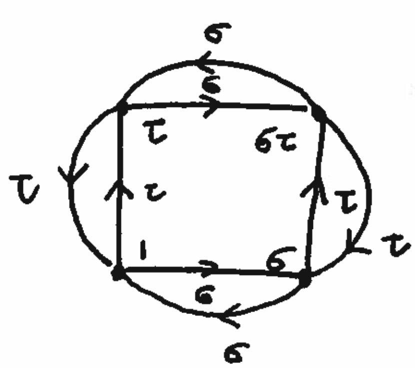
Def: Let S be a generating set for G. The Cayley graph $\Gamma(G,S)$ has

① a vertex v_g for each $g \in G$

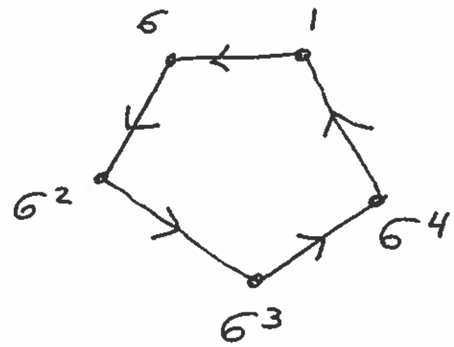
② an edge labled s from v_g to v_{gs} $\forall g \in G, s \in S$.

Ex: $G = C_2 \times C_2 = \{1, \tau, \sigma, \sigma\tau\}$

$S = \{\tau, \sigma\}$

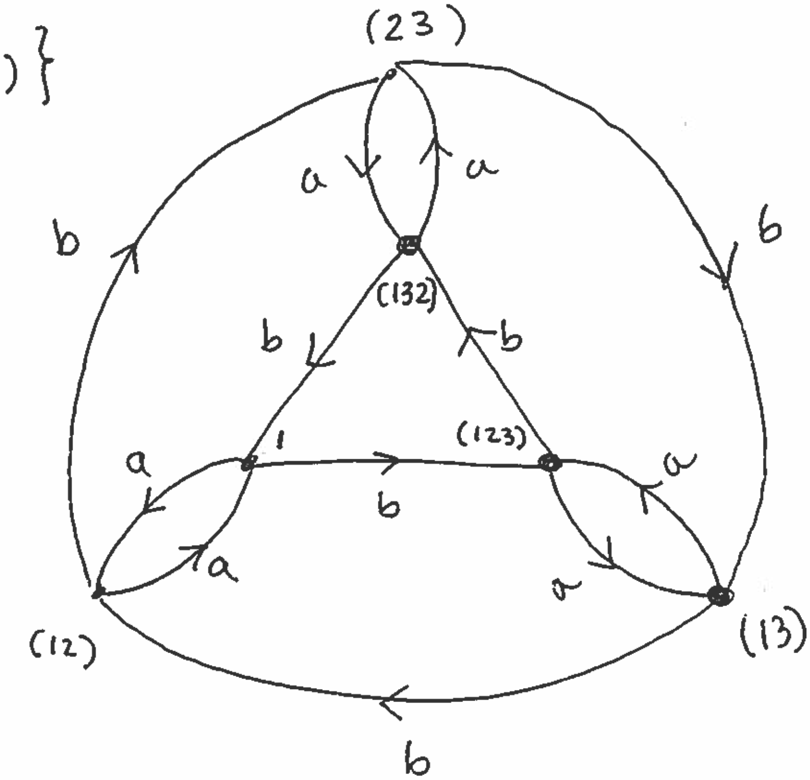


Ex: $G = C_n$, $S = \{\text{gen } \sigma\}$



Ex: $S_3 = \left\{ \begin{array}{l} 1, (12), (13), (23), \\ (123), (132) \end{array} \right\}$

$S = \{a = (12),\ b = (123)\}$



g joined to gs:

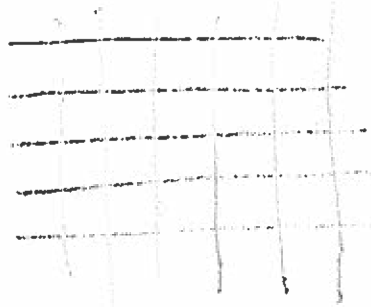
$(12)(123) = (1)(23)$

Q: What is $abab^{-1}ab$?

A: $a = (12)$

For any (G, S), the group G acts on $\Gamma(G, S)$ by $g \cdot V_h = V_{gh}$. This respects the edges, since an "s" edge joins $V_h \longmapsto V_{hs}$ and so there is also an "s" edge from $g \cdot V_h = V_{gh}$ to $g \cdot V_{hs} = V_{ghs}$.

Aside: • Can also do for infinite groups, leading to geometric group theory:



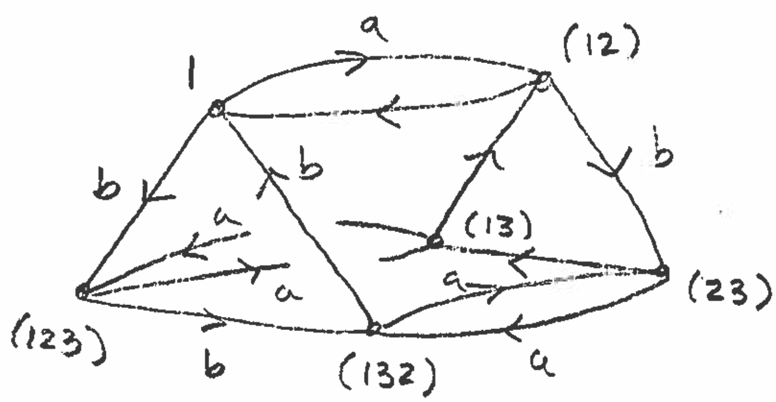
$G = \mathbb{Z}^2$

$S = \left\{ \begin{matrix} (0,1) \\ (1,0) \end{matrix} \right\}$

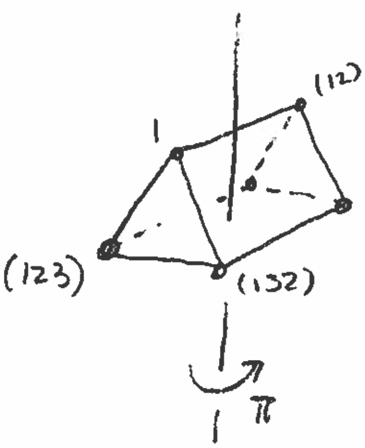
• Certain families of Cayley graphs are expanders:

$$G = PSL_2 \mathbb{F}_p \qquad S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

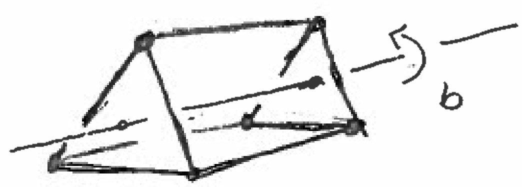
In the main example:



a acts on Γ by rotation by π:



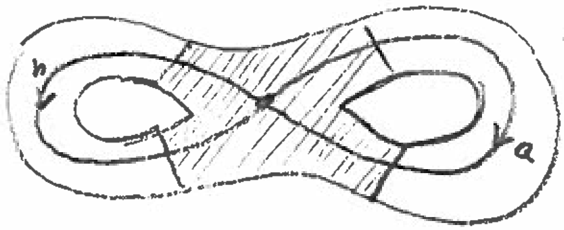
b acts on Γ by rotation by $2\pi/3$



ex: $(12)(132) = (13)$

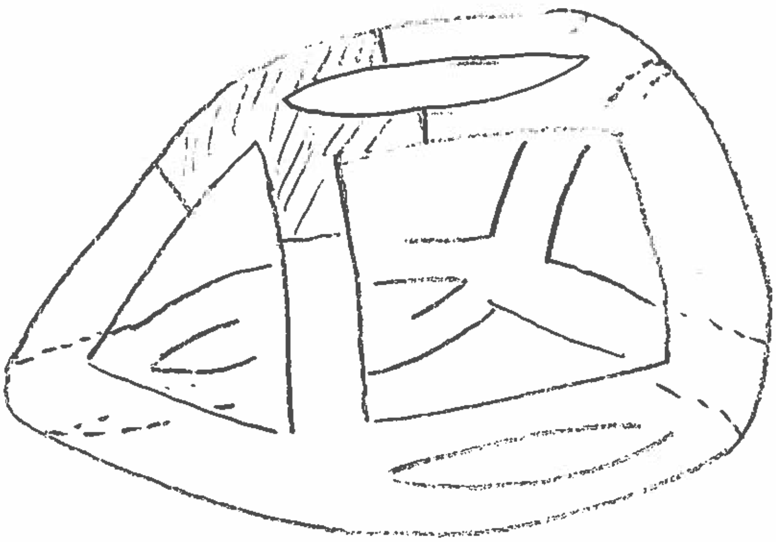
$(12)(123) = (23)$

What is Γ/G? A: 

As we want G to act on a surface, thicken Γ/G to 

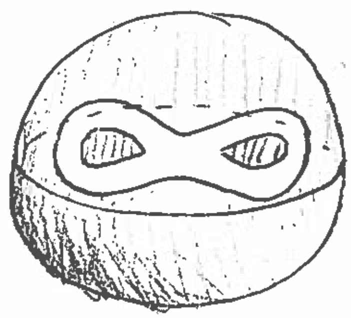
and

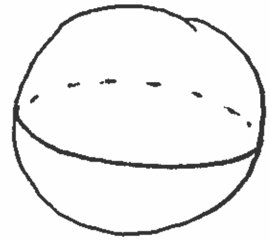
correspondingly thicken Γ to



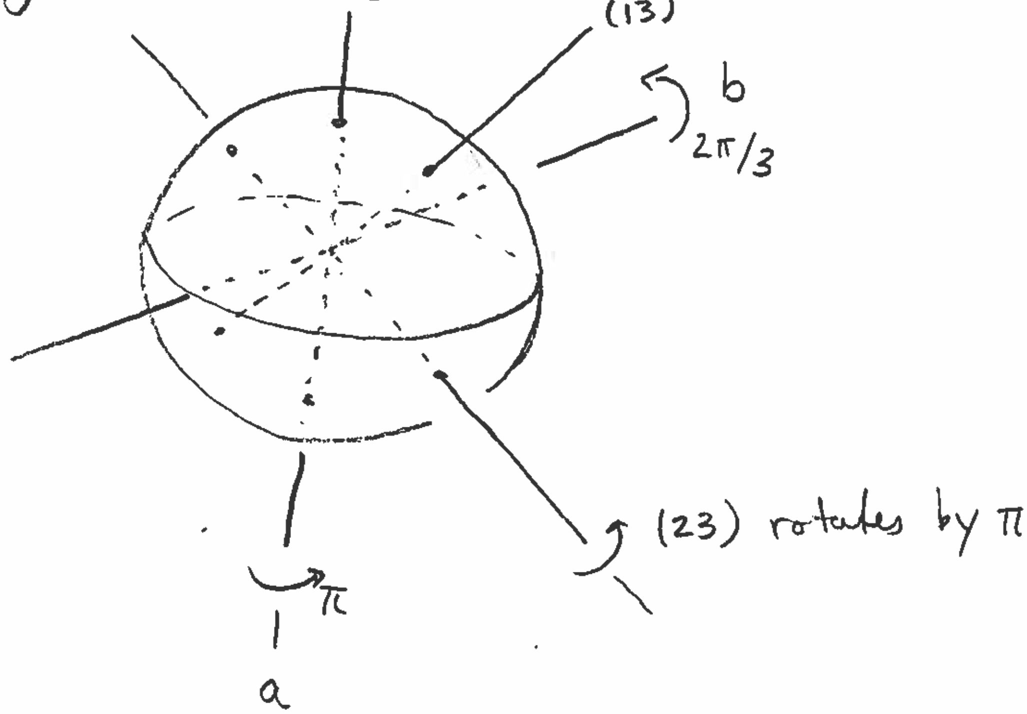
For each boundary circle, add a disc.

Γ/G becomes

$X =$  $= \mathbb{P}^1_{\mathbb{C}}$

and Γ becomes $Y =$  as well

The action of G on Γ gives an action of G on Y.

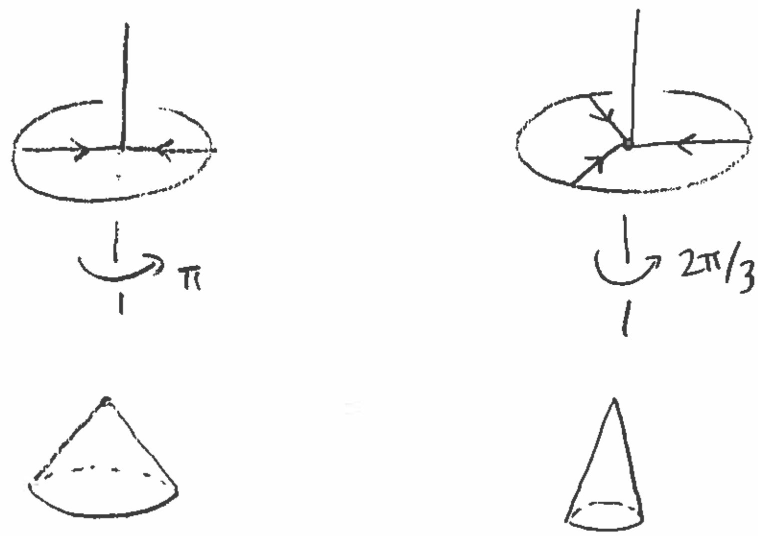


[S_3 = orientation-preserving symmetries of the bipyramid. ]

Have $p : Y \longrightarrow Y/_G = X$ extending $\Gamma \longrightarrow \Gamma/_G$.

First, note that $\Gamma \longrightarrow \Gamma/_G$ is locally 1-1 (a homeomorphism). The same is true for $p : Y \longrightarrow X$, except at the 8 points that are fixed by some elt of G, which are the centers of the added discs.

At these points, the quotient map p looks like



$z \longmapsto z^2$ $\qquad$ $z \longmapsto z^3$

So $p : Y \to X$ is a branched covering map which looks locally like a polynomial.

Next time: We will invoke the Riemann existence theorem to turn this into an honest rat'l map $Y \to X$, giving an extension $K/\mathbb{C}(t)$ with

Galois group S_3.

<u>Note</u>: The construction of $p: Y \longrightarrow X = \mathbb{P}^1_{\mathbb{C}}$ from $\Gamma(G, S)$ is completely general.

It's the Riemann existence theorem that's hard...