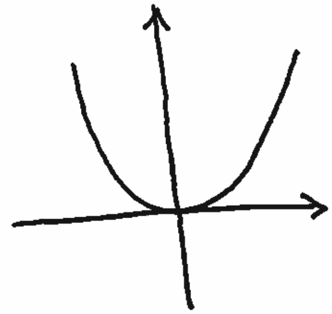
Lecture 39:



Ex from last time: $\mathbb{V}(y-x^2) = V \subseteq \mathbb{C}^2$

Consider $h: V \xrightarrow[\text{projection}]{} \{y\text{-axis}\}$, i.e. $h(x,y) = y$ in $\mathbb{C}[V]$. Gives a ring homom $h^*: \underset{\mathbb{C}[\mathbb{C}]}{\overset{\parallel}{\mathbb{C}[t]}} \longrightarrow \mathbb{C}[V]$

by $h^*(f(t)) = f(h(x,y)) = f(y)$. So $h^*(t) = y$.

Get a 1-1 field homomorphism $h^*: \mathbb{C}(t) \hookrightarrow \mathbb{C}(V)$

$$t \longmapsto y$$

Set $F = h^*(\mathbb{C}(t)) = \mathbb{C}(y)$ and $K = \mathbb{C}(V)$.

The field extension K/F is

1. Simple as $K = F(x)$.
2. Algebraic as x is a root of $z^2 - y \in F[z]$.
3. $z^2 - y$ is irred. in $F[z]$ by Eisenstein with $R = \mathbb{C}[y]$, $I = (y)$.

So

$$K = F[z]/(z^2-y) = F(\sqrt{y}).$$

Fun Fact: As abstract fields, $K \cong F$. Specifically, if we project onto the x-axis instead by

$g(x,y) = x$, we get $\mathbb{C}[t] \longrightarrow \mathbb{C}[V]$ which

$t \longmapsto x$

is onto since $y = x^2$ in $\mathbb{C}[V]$. Thus we get an <u>isomorphism</u> of fields $\mathbb{C}(t) \xrightarrow[g^*]{} \mathbb{C}(V)$.

Not as wierd as it seems: Note $\mathbb{C}(t^2) \subseteq \mathbb{C}(t)$ but $\mathbb{C}(t) \longrightarrow \mathbb{C}(t^2)$ is an isom.

$t \longmapsto t^2$

Same reasoning shows in general:

<u>Thm</u>: $V = \mathbb{V}(f) \subseteq \mathbb{C}^2$ an irreducible, smooth plane curve. Then $\mathbb{C}(V)$ is a finite extension of $\mathbb{C}(t)$.

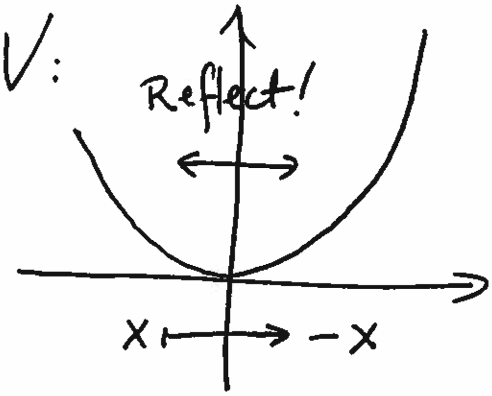
This has a partial converse:

<u>Thm</u>: Suppose K is a finite extension of $\mathbb{C}(t)$. Then $\exists$ an irreducible smooth curve $V \subseteq \mathbb{C}^n$ where $\mathbb{C}(V) = K$.

[Such fields are called function fields.]

Back to the example: K/F is Galois with group $G = \mathbb{Z}/2\mathbb{Z}$ whose gen sends $x = \sqrt{y} \longmapsto -x = -\sqrt{y}$.

This corresponds to a symmetry of V:

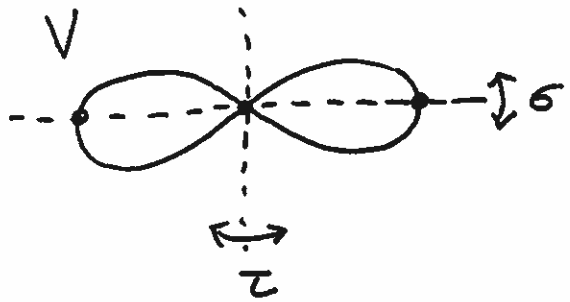


<u>Goal</u>: Given a finite group G, build $K/\mathbb{C}(t)$ with Galois group G.

<u>Outline</u>: [Reverse the above.]

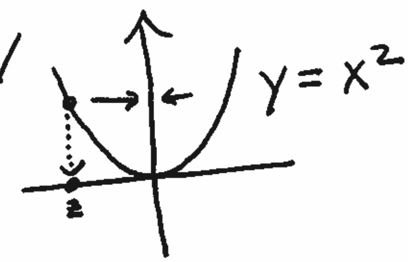
1. Given G, find a curve V (in $\mathbb{C}^n$ or $\mathbb{P}^n_{\mathbb{C}}$) where G acts as a group of symmetries of V.
2. Each $\sigma \in G$ gives an automorphism of $\mathbb{C}(V)$. [Think of $\mathbb{C}(V)$ as functions on V]
3. Identify $\mathbb{C}(V)_G$ with $\mathbb{C}(V/G)$ where V/G is the quotient, which is an alg. curve.
4. Do 1 so that $V/G = \mathbb{P}^1_{\mathbb{C}}$ and hence $\mathbb{C}(V/G) = \mathbb{C}(t)$. Thus we have an extension $\mathbb{C}(V)/\mathbb{C}(t)$ with Galois group G.

Thinking about 3:



V/G

$G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



Back to example:

Symmetry: $x \to -x$

$$V \xrightarrow{h} \mathbb{C}$$
$$(x,y) \longmapsto y$$

If we identify V with $\mathbb{C}$ by projection onto the x-axis, the map h becomes $\mathbb{C} \to \mathbb{C}$, $z \longmapsto z^2$

Let $\overline{V} \subseteq \mathbb{P}^2_{\mathbb{C}}$ be the corr. proj. curve. Have $\overline{V} \cong \mathbb{P}^1_{\mathbb{C}} \cong$ . We want to consider the corr.

map $\overline{h} : \overline{V} = \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$.

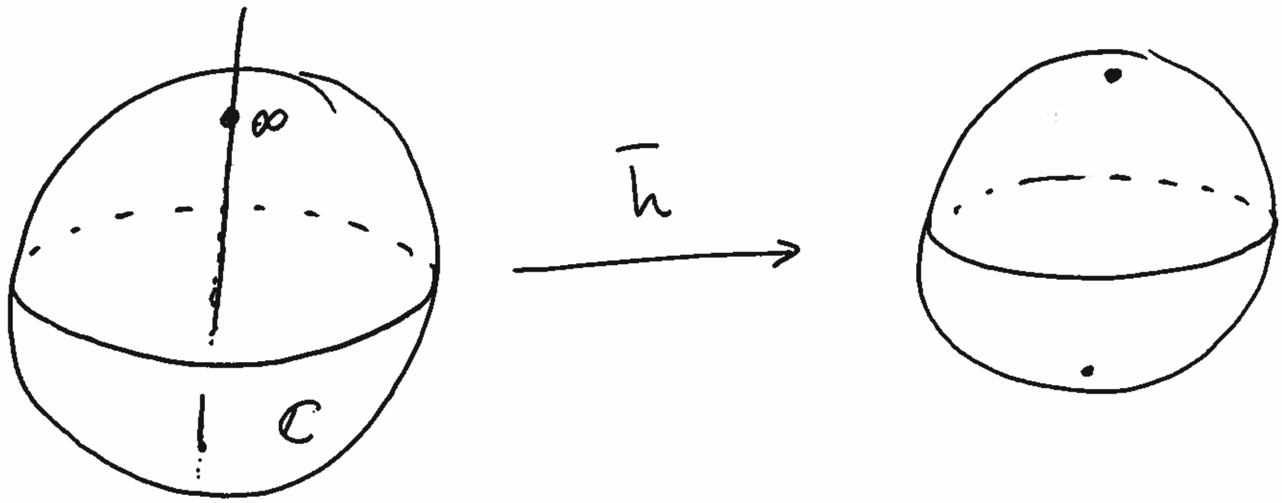
$$z \longmapsto z^2 \text{ for } z \in \mathbb{C}$$
$$\infty \longmapsto \infty$$

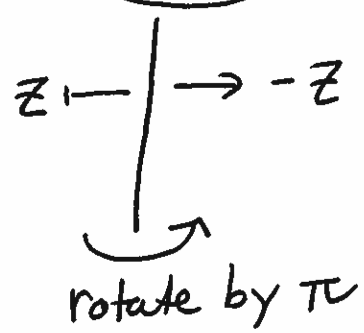
This is a polynomial map since $\mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$

$$(u:v) \longmapsto (u^2:v^2)$$

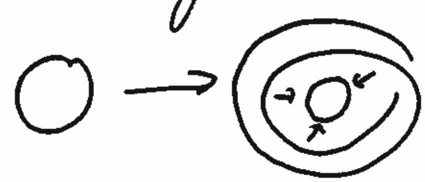
restricts to $\mathbb{C}^2$ as $u \mapsto u^2$ and sends $(1:0) \longmapsto (1:0)$.

What does it look like? First note $\bar{h}$ sends $(u:v)$ and $(-u:v)$ to the same pt. In pictures



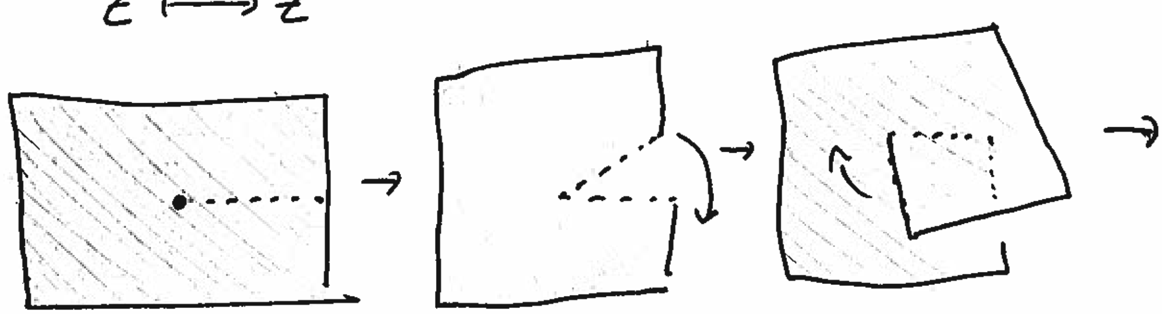


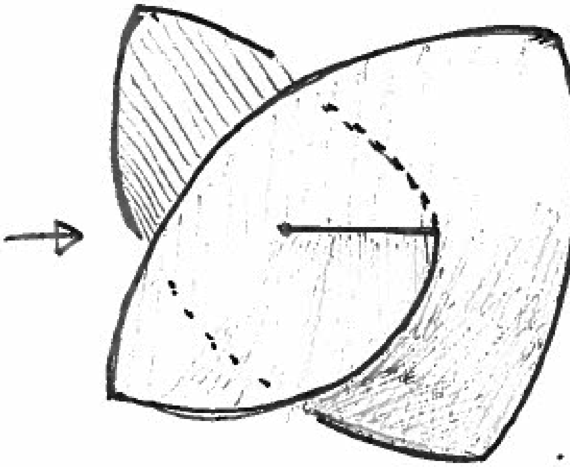
Map on the equator looks like



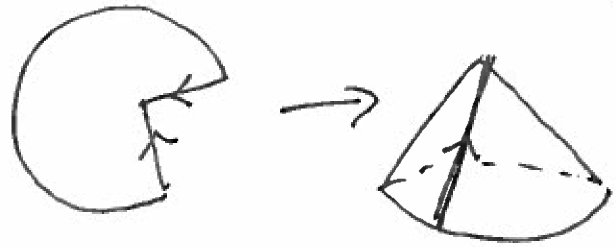
Compare: $\mathbb{C} \longrightarrow \mathbb{C}$ $\quad z = re^{i\theta} \longmapsto r^2 e^{i2\theta}$

$$z \longmapsto z^2$$

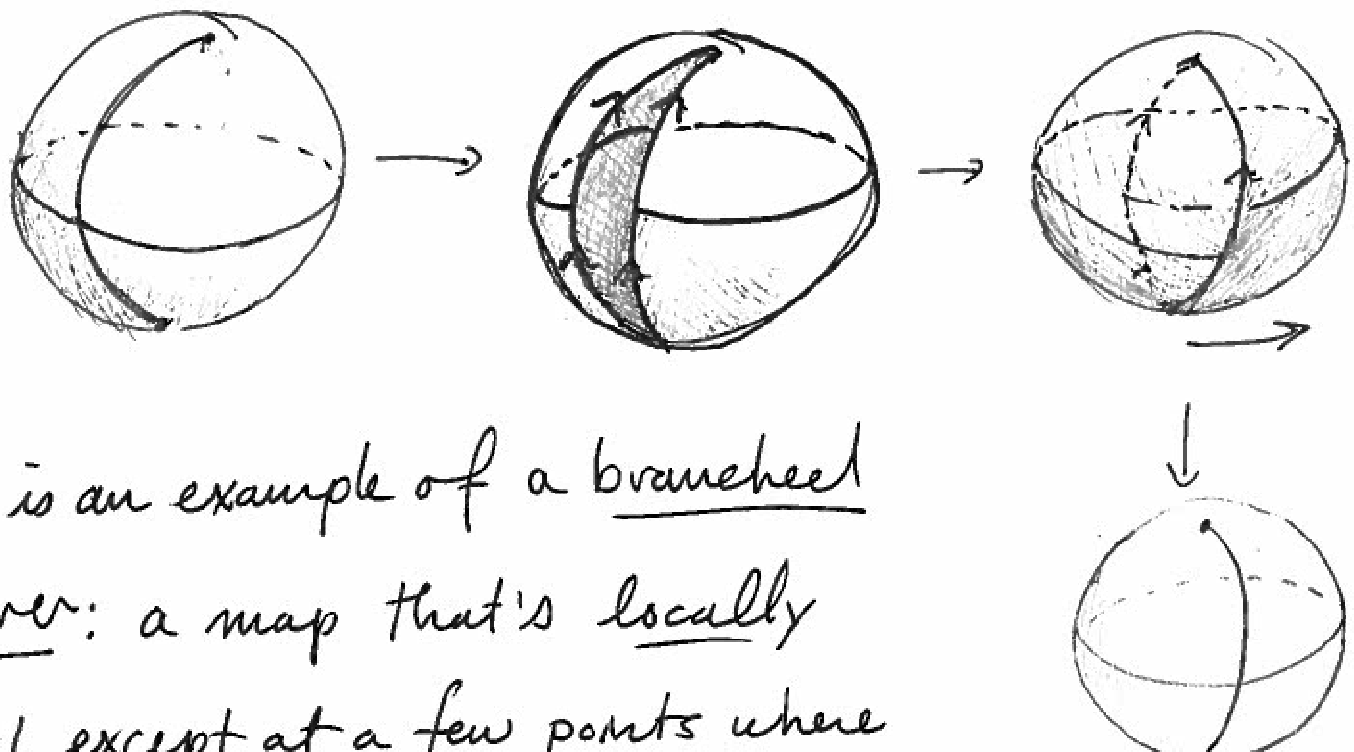




This is like a cone, but there is too much angle around the cone pt.



On $\mathbb{P}^1_{\mathbb{C}}$, have



h is an example of a branched cover: a map that's locally 1-1 except at a few points where it looks like $z \longmapsto z^n$.