Lecture 33: Functions on varieties.

Nullstellensatz: k alg. closed, $R = k[x_1, \dots, x_n]$
For all $I \subseteq R$ have $\mathbb{I}(\mathbb{V}(I)) = \mathrm{rad}(I)$. Moreover

$$\{\text{varieties in } k^n\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \{\text{radical ideals in } R\}$$

are inverse bijections. Also, every maximal ideal $I \subseteq R$ is of the form $(x_1 - a_1, \dots, x_n - a_n) = \mathbb{I}(\{a\})$ for some $a \in k^n$.

Hence $$\{a \in k^n\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \{\text{maximal ideals in } R\}$$

Hilbert's Basis Thm: For any field k, $R = k[x_1, \dots, x_n]$ is Noetherian

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Cor of basis theorem: Every ideal $I \subseteq R$ is finitely generated, i.e. $\exists f_1, \dots, f_k \in R$ with $I = (f_1, \dots, f_k)$

Note: This generalizes our old result that $k[x]$ is P.I.D.

Pf of Cor: If not, can choose $f_1, f_2, f_3, \dots \in I$ so that $f_{k+1} \notin I_k = (f_1, \dots, f_k)$. But then
$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \dots \text{ does not stabilize,}$$
violating that R is Noetherian. $\square$

Cor of Weak Null: k alg. closed, $V \subseteq k^n$ a variety.

Then there is a bijection

$$\{\text{pts } a \in V\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \left\{\begin{matrix}\text{maximal ideals} \\ I \supseteq \mathbb{I}(V)\end{matrix}\right\}$$

Pf. If $a \in V$, then $\mathbb{I}(a) \supseteq \mathbb{I}(V)$ and $\mathbb{I}(a)$ is maximal. Conversely, any max ideal $I \supseteq \mathbb{I}(V)$ is $\mathbb{I}(a)$ for some $a \in k^n$; as $\mathbb{V}$ reverses inclusion, have $a \in V$. $\square$

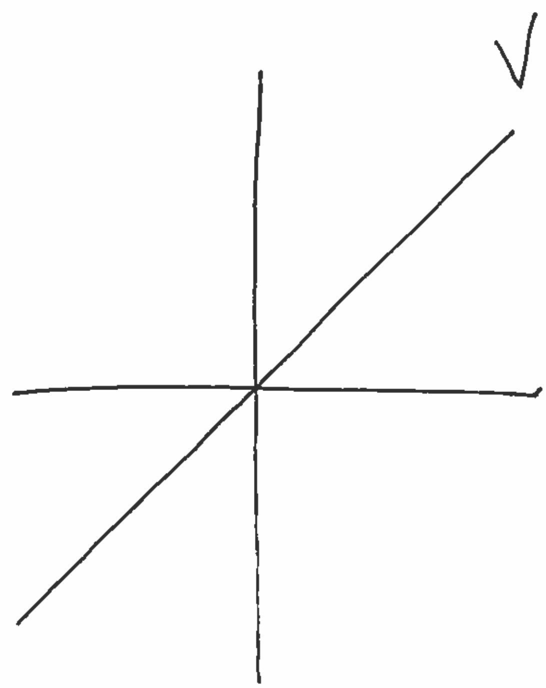
Functions of Varieties: $V \subseteq k^n$ a variety

Consider polynomial functions $f : V \longrightarrow k$.

Ex: $V = \mathbb{V}(x-y) \subseteq \mathbb{R}^2$

$f = x^2y + y + x$

$g = y^2x + 2y$



While these are different elements of $\mathbb{R}[x,y]$,

they give the same fn on V, namely $(x,y) \longmapsto x^3 + 2x$

since $y = x$ on V.

<u>Def</u>: The <u>coordinate ring</u> of V is

$$k[V] = \left\{ f : V \to k \;\middle|\; \begin{matrix} f \text{ is the restriction of} \\ \text{an elt of } k[x_1, \ldots, x_n] \end{matrix} \right\}$$

Note $k[V] = k[x_1, \ldots, x_n] / \mathbb{I}(V)$

since $\mathbb{I}(V)$ is exactly the kernel of the ring homomorphism. $k[x_1, \ldots, x_n] \longrightarrow k[V]$.

<u>Cor</u>: V is irreducible iff $k[V]$ is an integral domain.

<u>Pf</u>: Know V irred $\Leftrightarrow \mathbb{I}(V)$ is prime $\Leftrightarrow R/\mathbb{I}(V)$ is an int. domain. $\square$

<u>Cor</u>: If k is alg. closed, $V \subseteq k^n$ a variety, then

$$\{a \in V\} \xleftrightarrow[\text{bijection}]{} \{\text{maximal ideals in } k[V]\}$$

<u>Pf</u>: Max ideals in $R/\mathbb{I}(V)$ correspond to max. ideals $\mathbb{I}(V) \subseteq I \subseteq R$. $\square$

Can apply this philosophy more broady:

(X, d) compact metric space, e.g. closed subset of $\mathbb{R}^n$

$C(X) = \{f : X \to \mathbb{R} \text{ continuous}\}$ is a ring.

<u>Thm</u>: $\{a \in X\} \longrightarrow \left\{\begin{matrix}\text{maximal ideals}\\ \text{in } C(X)\end{matrix}\right\}$ is a bijection.

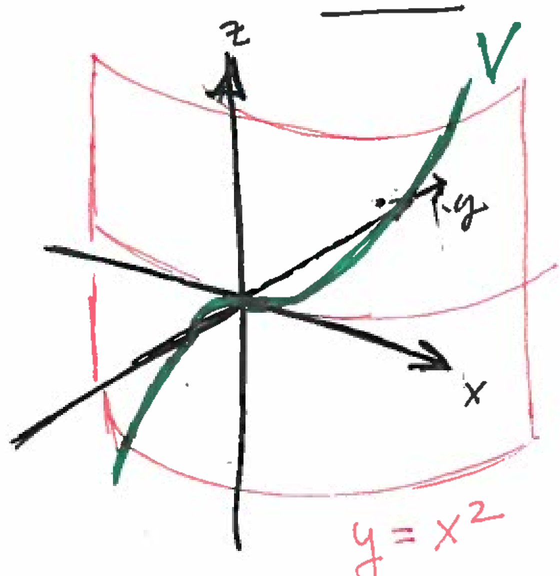
$$a \longmapsto I(a) = \{f \in C(X) \mid f(a) = 0\}$$

[Mumble about non-commutative geometry if time allows...]

<u>Example</u>: $V = \mathbb{V}(y - x^2, z - x^3) \subseteq \mathbb{R}^3$ The <u>twisted cubic</u>

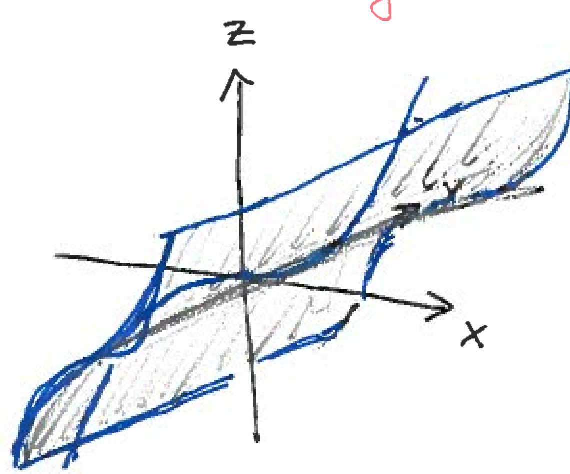
Now $\mathbb{R}[V] = \mathbb{R}[x,y,z]/\mathbb{I}(V)$

$\cong \mathbb{R}[x]$



<u>Reason</u>: $I = (y - x^2, z - x^3)$ $\subseteq \mathbb{I}(V)$, so $\mathbb{R}[V]$ is a quotient of $R/I \cong \mathbb{R}[x]$.

Suppose $f \neq g \in \mathbb{R}[x]$, I claim they are distinct in $\mathbb{R}[V]$.



Certainly, there exists $a \in \mathbb{R}$ with $f(a) \neq g(a)$.

Then $(a, a^2, a^3) \in V$ and so $f \neq g$ in $\mathbb{R}[V]$.

So $\mathbb{R}[V] \cong \mathbb{R}[x]$ and $\mathbb{I}(V) = (y - x^2, z - x^3)$

Q: What is another variety W with $\mathbb{R}[W] \cong \mathbb{R}[x]$?

A: $W = \mathbb{R}$ as $\mathbb{R}[W] = \mathbb{R}[t] / \underset{\underset{0}{\shortparallel}}{\mathbb{I}(W)} = \mathbb{R}[t]$

So are V and W the "same" somehow? Yes!

Consider $f: W \longrightarrow V$ which is a polynomial
$t \longmapsto (t, t^2, t^3)$

mapping. This is a bijection, and there's an inverse

poly. map $g: V \longrightarrow W$. These
$(x,y,z) \longmapsto x$

maps induce isomorphisms $\mathbb{R}[V] \underset{g^*}{\overset{f^*}{\rightleftarrows}} \mathbb{R}[W]$

Here $f^*(h: V \to \mathbb{R}) = h \circ f$, e.g. $xy + z \mapsto 2t^3$.