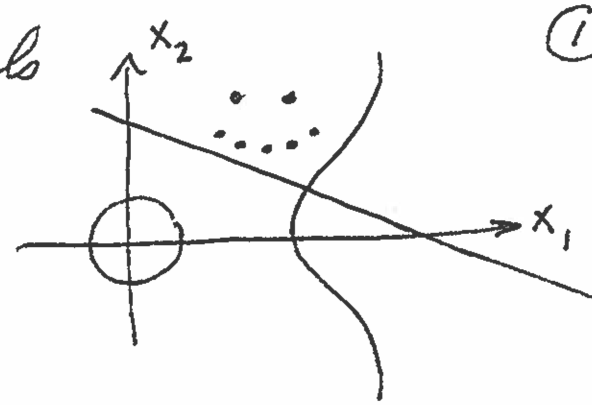
Lecture 31: Varieties and ideals



k = field

Affine space: k^n

$I \subseteq k[x_1, \dots, x_n]$

Algebraic Variety: $\mathbb{V}(I) = \{a \in k^n \mid f(a) = 0 \ \forall f \in I\}$

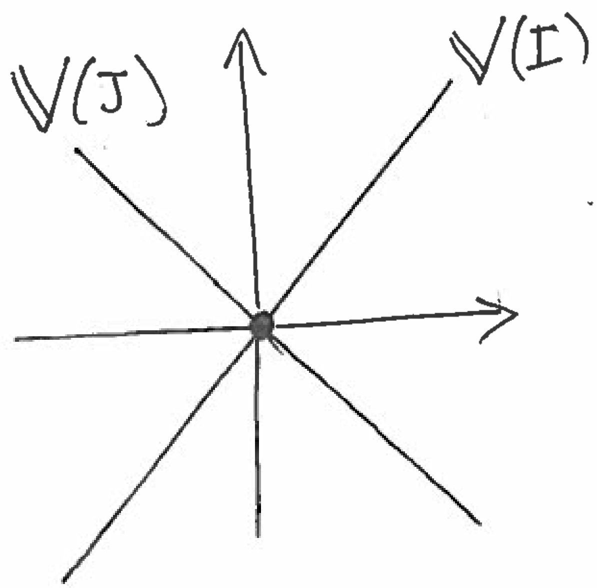
I might as well be an ideal.

Basic Props ① $I \subseteq J \Rightarrow \mathbb{V}(I) \supseteq \mathbb{V}(J)$

② $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I \cup J) = \mathbb{V}(I + J)$

Ex: $k = \mathbb{R}$, $n = 2$ $\quad I = (x-y) \quad J = (x+y)$

$I + J = (x, y)$



③ $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cdot J)$

Ex: $I \cdot J = ((x-y)(x+y))$

$= \{f(x,y)(x-y)(x+y)\}$

Let V be an algebraic variety. Set

$$\mathbb{I}(V) = \{ f \in k[x_1, \dots, x_n] \mid f(a) = 0 \text{ for all } a \in V \}$$

__Note__: $\mathbb{V}(\mathbb{I}(V)) = V$ since if $V = \mathbb{V}(I)$ then $\mathbb{I}(V) \supseteq I$ and every $f \in \mathbb{I}(V)$ vanishes on V.

__Key point__: $\mathbb{I}(\mathbb{V}(I)) \supseteq I$ but __not__ always equal!

__Ex__: $I = (x^2) \subseteq k[x]$

$\mathbb{V}(I) = \{0\}$ but $\mathbb{I}(\{0\}) = (x)$.

[In practice, this is a real problem...]

__Def__: For I an ideal in a (commutative) ring R, it's __radical__ is $\operatorname{rad}(I) = \{ a \in R \mid a^n \in I \}$

__Ex__: $\operatorname{rad}((x^2)) = (x)$ "Zero locus thm."

__Hilbert's Nullstellensatz__: Suppose k is alg. closed. Then $\mathbb{I}(\mathbb{V}(I)) = \operatorname{rad}(I)$ for all ideals $I \subseteq k[x_1, \dots, x_n]$. Moreover we have inverse bijections

$$\left\{ \begin{matrix} \text{Algebraic} \\ \text{varieties in } k^n \end{matrix} \right\} \underset{\mathbb{V}}{\overset{\mathbb{I}}{\rightleftarrows}} \left\{ \begin{matrix} \text{radical ideals} \\ \text{in } k[x_1, \dots, x_n] \end{matrix} \right\}$$

<u>Easy half of Pf</u>: $\mathbb{I}(\mathbb{V}(I)) \supseteq \mathrm{rad}(I)$

Suppose $f \in \mathrm{rad}(I)$, i.e. $f^n \in I$. If $a \in V(I)$, then $0 = f^n(a) = (f(a))^n \Rightarrow f(a) = 0$.
So $f \in \mathbb{I}(\mathbb{V}(I))$. ▨

Other half: next lecture.

<u>Ex</u>: $I = (x^2 - 2) \subseteq \mathbb{Q}[x]$. Then $\mathbb{I}(\mathbb{V}(I)) = \mathbb{Q}[x]$ since $\mathbb{V}(I) = \phi$.

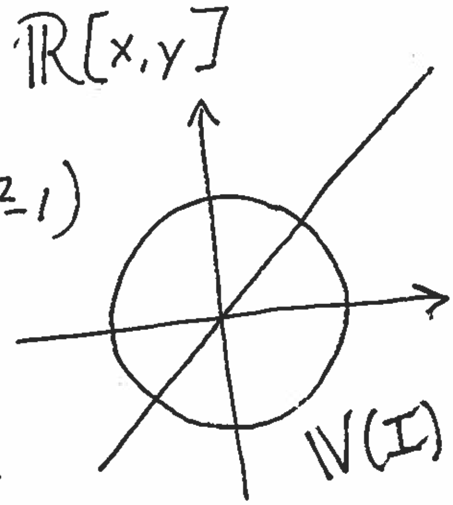
<u>Ex</u>: $I = (x^2+1) \subseteq \mathbb{R}[x]$
$\mathbb{I}(\mathbb{V}(I)) = \mathbb{R}[x]$.

<u>Nullstellensatz II</u>: $k \subseteq \bar{k}$ with $\bar{k}$ algebraically closed. If $I \subseteq k[x_1, \ldots, x_n]$, then

$$\mathbb{I}_k(\underbrace{\mathbb{V}_{\bar{k}}(I)}_{\subseteq \bar{k}^n}) = \mathrm{rad}(I).$$

Decomposing varieties:

$$I = (x^3 + xy^2 - yx^2 - y^3 - x + y) \subseteq \mathbb{R}[x,y]$$



Turns out $\mathbb{V}(I) = \mathbb{V}(x-y) \cup \mathbb{V}(x^2+y^2-1)$

and in fact $I = (x-y)(x^2+y^2-1)$.

Can $\mathbb{V}(x-y)$ also be written as a union of two varietres?

<u>Def</u>: A variety V is <u>irreducible</u> if whenever $V = V_1 \cup V_2$ for varieties V_i, then $V = V_1$ or $V = V_2$.

<u>Thm</u>: V is irreducible iff $\mathbb{I}(V)$ is prime.

<u>Proof</u>: ($\Longrightarrow$) Suppose $f_1 \cdot f_2 \in \mathbb{I}(V)$. Set

$$V_i = V \cap \mathbb{V}(f_i) = \mathbb{V}(\mathbb{I}(V) + (f_i))$$
$$= \{\text{points of } V \text{ where } f_i = 0\}$$

For $a \in V$, we have $(f_1 \cdot f_2)(a) = f_1(a) \cdot f_2(a) = 0$

$\Rightarrow f_1(a) = 0$ or $f_2(a) = 0$. So $V = V_1 \cup V_2$.

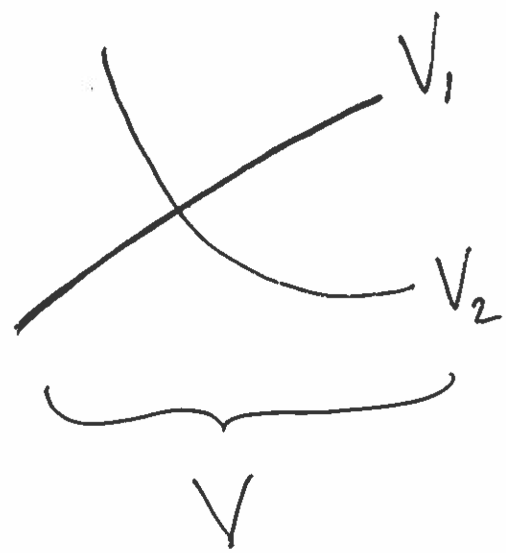
As V is irreducible, must have one $V_i = V$, say V_1.

Thus $f_1(a) = 0$ for all $a \in V \Rightarrow f_1 \in \mathbb{I}(V)$. So $\mathbb{I}(V)$ is prime.

($\Leftarrow$) Suppose $V = V_1 \cup V_2$. Assume $V \neq V_1$.

As $V_1 \subsetneq V$, have $\mathbb{I}(V_1) \supsetneq \mathbb{I}(V)$.

[Apply $\mathbb{V}$ and use $\mathbb{V}(\mathbb{I}(V)) = V$]



Pick $f_1 \in \mathbb{I}(V_1) \setminus \mathbb{I}(V)$.

Suppose $f_2 \in \mathbb{I}(V_2)$. Then

$f_1 f_2 = 0$ on $V \Rightarrow f_1 f_2 \in \mathbb{I}(V)$.

As $\mathbb{I}(V)$ is prime, must have one $f_i \in \mathbb{I}(V)$ which must be f_2. Hence $\mathbb{I}(V_2) \subseteq \mathbb{I}(V)$ and so $V_2 \supseteq I \Rightarrow V = V_2$. So V is irreducible.

$\square$