

Math 418: Takehome 2 due Wednesday, April 22, 2015.

Disclaimer, Terms, and Conditions: You may not discuss the exam with anyone except myself. You may *only* consult the following:

- The beloved(?) text, Dummit and Foote's *Abstract Algebra*.
- Cox, Little, and O'Shea, *Ideals, Varieties, and Algorithms*.
- Your class notes and returned HW sets.
- My online class notes and HW solutions.

You can use any result in chapters 14-15 of Dummit and Foote, even if I didn't cover it in class. You can also use the result of any HW problem that was assigned, whether or not you did it. While I believe all the questions are stated correctly, please contact me if you think something is fishy.

Office hours: While discussion of these specific problems will be limited to clarifying their statements, I will be happy to answer any broader questions about the course material during my usual office hours (Monday and Tuesday from 2:30–3:30pm and also by appointment).

1. Consider the cyclotomic field $K = \mathbb{Q}(\zeta_n)$ and let $G = \text{Gal}(K/\mathbb{Q})$.
 - (a) Let $\tau \in G$ be the restriction of complex conjugation. Find the element that τ corresponds to under the isomorphism $G \cong (\mathbb{Z}/n\mathbb{Z})^\times$.
 - (b) Let $K^+ = K \cap \mathbb{R}$. Prove that $K^+ = \mathbb{Q}(\alpha)$ where $\alpha = \zeta_n + \zeta_n^{-1}$.
2. Find the Galois group of $x^4 - 7$ over \mathbb{Q} explicitly as a permutation group on the roots

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{\sqrt[4]{7}, -\sqrt[4]{7}, \sqrt[4]{7}i, -\sqrt[4]{7}i\}$$

Clarification: Your answer should both give the isomorphism type of $\text{Gal}(K/\mathbb{Q})$ and identify the explicit subgroup of $S_4 = \text{SymmetricGroup}(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})$ which is its image under the map $\text{Gal}(K/\mathbb{Q}) \rightarrow S_4$.

3. In a field F of characteristic 2, consider an irreducible cubic $f(x) = x^3 + ax^2 + bx + c \in F[x]$.
 - (a) Prove that the discriminant D of f is always a square in F .
 - (b) Let K be the splitting field of f over F . Prove that K is also the splitting field of an irreducible polynomial of the form $x^3 + px + q \in F[x]$.
 - (c) For $f(x) = x^3 + px + q$, consider the polynomial $g(x) = x^2 + qx + (p^3 + q^2)$. Prove that $\text{Gal}(K/F)$ is A_3 if $g(x)$ has a root in F and S_3 otherwise.

More problems on back of sheet!

4. Let $R = \mathbb{C}[x_1, \dots, x_n]$.
- (a) If $f \in R$ is an irreducible polynomial prove that the variety $\mathbf{V}(f) \subset \mathbb{C}^n$ is irreducible.
 - (b) If $I \subset R$ is a proper ideal, prove that its radical is the intersection of all maximal ideals containing I .
5. Let V be an algebraic variety in k^n , and set $I = \mathbf{I}(V)$. Recall the coordinate ring of V is

$$\begin{aligned} k[V] &= \{f: V \rightarrow k \mid f \text{ is the restriction of a polynomial in } k[x_1, \dots, x_n].\} \\ &= k[x_1, \dots, x_n]/I \end{aligned}$$

In particular, two elements of $k[V]$ are the same if they agree at every point of V , even if nominally they come from different polynomials.

The ring $k[V]$ contains k as the subring of constant functions, coming from the polynomials with only a constant term. Thus it is a vector space over k . Prove that the following are equivalent.

- (a) V is a finite set of points in k^n .
- (b) $k[V]$ is finite-dimensional as a k -vector space.

Hints: For (a) \Rightarrow (b) consider $k[V]$ as a subspace of the vector space of *all* functions $f: V \rightarrow k$. For (b) \Rightarrow (a), for each coordinate, try to show that the set $\{a_i \mid (a_1, \dots, a_n) \in V\}$ is finite.