

## Math 418: Takehome 1 due Wednesday, February 18, 2015.

**Disclaimer, Terms, and Conditions:** You may not discuss the exam with anyone except myself. You may *only* consult the following:

- The beloved(?) text, Dummit and Foote's *Abstract Algebra*.
- Your class notes and returned HW sets.
- My online class notes and HW solutions.

You can use any result in Chapters 7–9 and Sections 13.1–13.2 of Dummit and Foote, even if I didn't cover it in class. You can also use the result of any HW problem that was assigned, whether or not you did it.

**Office hours:** While discussion of these specific problems will be limited to clarifying their statements, I will be happy to answer any broader questions about the course material during my usual office hours (Monday and Tuesday from 2:30–3:30pm), extra office hours (Friday Feb 13, 1:00–2:30pm), or by appointment.

1. Let  $F$  be a field. Consider the ring  $R = F[[t]]$  of *formal power series* in  $t$ , namely things of the form

$$\sum_{n=0}^{\infty} a_n t^n = a_0 + a_1 t + a_2 t^2 + \cdots \quad \text{where } a_n \in F.$$

Here “formal” means the above “sum” is really just an infinite list of elements of  $F$ ; there's no notion of convergence involved. Elements of  $R$  are added term by term, and multiplication is as if they were polynomials. More precisely

$$\sum_{n=0}^{\infty} a_n t^n \times \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) t^n$$

It is clear that  $R$  is a commutative ring with unit.

- (a) Prove that  $\alpha$  in  $R$  is a unit if and only if the constant term  $a_0 \neq 0$ . (Example: The inverse of  $1 - t$  is  $1 + t + t^2 + t^3 + t^4 + \cdots$ .)
- (b) Prove that  $R$  is a Euclidean domain with respect to the norm  $N(\alpha) = n$  if  $a_n$  is the first term of  $\alpha$  that is non-zero. (If  $F = \mathbb{C}$  and the power series converges near  $t = 0$ , then this norm is just the order of zero of the corresponding function at 0.)
- (c) In the polynomial ring  $R[x]$ , prove that  $x^n - t$  is irreducible.

2. Let  $R = \mathbb{Z}[i]$ .

(a) Prove that  $R/(1+i)$  is a field of order 2.

(b) Let  $\pi \in R$  be irreducible. Consider the ideals  $I_n = (\pi^n)$ . Prove that  $R/(\pi) \cong I_n/I_{n+1}$  as additive abelian groups. Hint: the isomorphism is multiplication by  $\pi^n$ .

(c) Again for irreducible  $\pi$ , prove that  $|R/(\pi^n)| = |R/(\pi)|^n$ . Here  $|\cdot|$  denotes the number of elements in a finite set. (This is a key step in proving that for *any*  $\pi \in R$  that  $|R/(\pi)| = N(\pi) = |\pi|^2$ .)

(d) For  $\pi = 1+i$ , are  $R/(\pi^3)$  and  $\mathbb{Z}/8\mathbb{Z}$  isomorphic as rings?

3. Section 13.2, #8.

4. Section 13.2, #13.

5. Section 13.2, #20.