

## Math 418: HW 8 due Wednesday, April 15, 2015.

Webpage: <http://dunfield.info/418>

Office hours: Monday and Tuesday from 2:30–3:30pm and by appointment.

1. Fix a prime  $p$ . Show that the following subgroup of  $\mathrm{GL}_2\mathbb{F}_p$  is solvable:

$$B = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \mid x, y \in \mathbb{F}_p^\times, z \in \mathbb{F}_p \right\}$$

Here, the group operation is just matrix multiplication.

2. (a) Prove directly from the definition that  $S_4$  is solvable.  
(b) Prove that  $A_5$  is simple using the following outline.
  - (i) Show  $A_5$  has 5 distinct conjugacy classes of elements, and count the number of elements in each class.
  - (ii) For any normal subgroup  $H \triangleleft G$  show that  $H$  is a union of conjugacy classes of  $G$ .
  - (iii) If  $N \triangleleft A_5$  use that  $|N|$  divides  $|A_5|$  and parts (i) and (ii) to show that  $N = \{1\}$  or  $A_5$ .

Alternatively, give a geometric proof using the fact that  $A_5$  is the group of Euclidean isometries of a regular dodecahedron.

Remark:  $A_5$  is the smallest of all the simple groups. In fact, every group of order less than 60 is solvable.

- (c) Use (b) to show that  $S_n$  is not solvable for  $n \geq 5$ .
3. (Section 14.7, #12) Let  $L$  be the Galois closure of a finite extension  $\mathbb{Q}(\alpha)$  over  $\mathbb{Q}$ . If  $p$  is a prime dividing the order of  $\mathrm{Gal}(L/\mathbb{Q})$ , show that there is a subfield  $F$  of  $L$  with  $[L:F] = p$  and  $L = F(\alpha)$ .  
Hint: You'll need to use Theorem 18 from Section 4.5: if  $p$  is a prime dividing the order of a finite group  $G$ , then  $G$  has an element of order  $p$ .

4. (Section 14.7, #13) Let  $F \subset \mathbb{R}$  be a field. Let  $a$  be an element of  $F$  which has a real  $n^{\mathrm{th}}$  root  $\alpha = \sqrt[n]{a}$ , and set  $K = F(\alpha)$ . Prove that if  $L$  is any Galois extension of  $F$  contained in  $K$  then  $[L:F] \leq 2$ .

5. For a field  $k$ , here are some basic problems for varieties in  $k^2$ , where we take the coordinates to be  $(x, y)$ . Except for part (b), *assume that  $k$  is algebraically closed*.

- (a) Let  $V$  be the  $x$ -axis, i.e.  $V = \mathbf{V}(y)$ . Prove that  $V$  is irreducible. Hint: Show a prime ideal is radical.
- (b) Give an example of a field  $k$ , necessarily not algebraically closed, for which the  $x$ -axis is *reducible*.
- (c) Prove that  $V = \mathbf{V}(x - y)$  is irreducible.
- (d) Prove that  $S = \{(a, a) \in k^2 \mid a \neq 1\}$  is *not* an algebraic variety if  $k = \mathbb{C}$ .
- (e) What is the decomposition of  $V = \mathbf{V}(x^2 - y^2)$  into irreducibles? **Warning:** The answer depends on  $k$ !