Lecture 6: More on cup products

Last time: R a ring, X a space. The cup product

$$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\cup} H^{k+\ell}(X;R)$$

is induced by

$$\begin{array}{ccccc} C^k(X;R) & \times & C^\ell(X;R) & \longrightarrow & C^{k+\ell}(X;R) \\ \alpha & & \beta & & \alpha \cup \beta \end{array}$$

where for $\sigma : \Delta^{k+\ell} \longrightarrow X$ we have

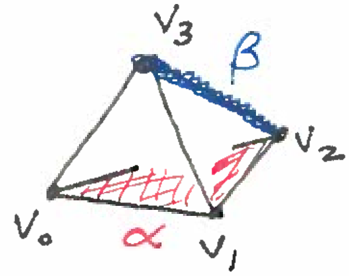
$$\alpha \cup \beta(\sigma) = \alpha(\sigma|_{[v_0,\dots,v_k]})\,\beta(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

Thm: $\alpha \in H^k(X)$, $\beta \in H^k(X)$ then $\alpha \cup \beta = (-1)^{k\ell} \beta \cup \alpha$

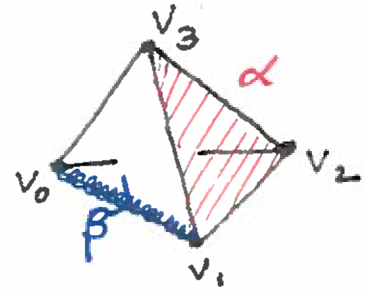
Cor: For k odd, $\alpha \cup \alpha = -\alpha \cup \alpha \Rightarrow 2(\alpha \cup \alpha) = 0$ in $H^{2k}(X)$. So if H^{2k} has no 2-torsion then $\alpha \cup \alpha = 0$.

[**Cor.** If cohomology is all in even dim, then H^* is commutative.
(or if $R = \mathbb{Z}/2$)]

Issue:
$k=2,\ \ell=1$



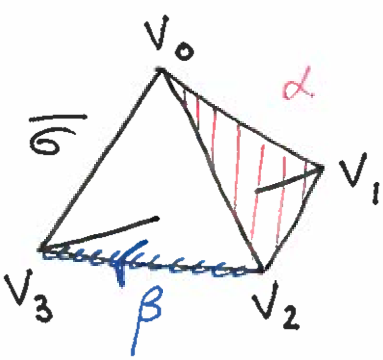
vs.



$\alpha \cup \beta(\sigma) = \alpha(\text{front})\, \beta(\text{back})$ $\qquad$ $\beta \cup \alpha(\sigma) = \beta(\text{front})\, \alpha(\text{back})$

similar

$$\bar{\sigma} = \sigma \circ \left(\begin{array}{l} \text{linear map } \tau \\ [v_0, v_1, \dots, v_n] \longrightarrow [v_n, v_{n-1}, \dots, v_0] \end{array} \right)$$



$$\alpha \cup \beta(\bar{\sigma}) = \alpha(\text{front } \bar{\sigma})\, \beta(\text{back of } \bar{\sigma})$$

Set $\varepsilon_n = (-1)^{\frac{n(n+1)}{2}}$ [whether τ preserves/reverses orientation.]

and define $\rho: C_n(X) \longrightarrow C_n(X)$ by $\rho(\sigma) = \varepsilon_n \bar{\sigma}$

Props:
- ρ is a chain map, chain homotopic to id.
- ρ^* on C^* induces id on $H^*(X)$
- $\varepsilon_{k+\ell}\, \rho^*(\alpha \cup \beta) = \varepsilon_k \varepsilon_\ell\, \rho^*(\beta) \cup \rho^*(\alpha)$

[See text for details.]

Hence in H^* have
$$\begin{aligned} \alpha \cup \beta &= \varepsilon_{k+\ell} \cdot \varepsilon_k \cdot \varepsilon_\ell \cdot \beta \cup \alpha \\ &= (-1)^{k \cdot \ell}\, \beta \cup \alpha. \end{aligned}$$

$\underline{Ex}$: $T^n = S^1 \times \dots \times S^1 = \mathbb{R}^n / \mathbb{Z}^n$

$$\begin{array}{c} S^1 \times \dots \\ \downarrow p_i \\ S^1 \end{array}$$

Fix a gen α of $H^1(S^1; R)$ and set $\alpha_i = p_i^*(\alpha)$.
$\underset{R}{\cong}$

$H^*(T^n)$ = exterior algebra on $\alpha_1, \dots, \alpha_n$, where $\alpha_i \cup \alpha_j = -\alpha_j \cup \alpha_i$

Specifically:
$H^k(T^n)$ = free R-module with basis $\alpha_{i_1} \cup \dots \cup \alpha_{i_k}$ where $i_1 < i_2 < \dots < i_k$

Recall the cross product: $H^k(X) \times H^\ell(Y) \xrightarrow{\times} H^{k+\ell}(X \times Y)$

$$\beta \quad \gamma \longmapsto p_X^*(\beta) \cup p_Y^*(\gamma)$$

In our context with $n = 4$:

$$\alpha_1 \cup \alpha_3 = \alpha \times 1 \times \alpha \times 1 \qquad \alpha_1 \cup \alpha_2 = \alpha \times \alpha \times 1 \times 1$$

$\underline{Thm}$: $H^k(T^n)$ is the free R module on $\{c_1 \times \dots \times c_n \mid c_i = 1 \text{ or } \alpha\}$

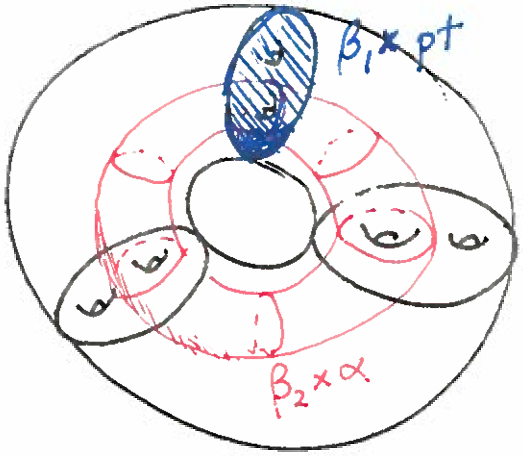
Follows inductively from:

$\underline{Lemma}$: $H^{n+1}(Y) \times H^n(Y) \longrightarrow H^{n+1}(Y \times S^1)$ is an isom.

$$(\beta_1, \beta_2) \longmapsto \beta_1 \times 1 + \beta_2 \times \alpha$$

[Every cocycle is a sum of two product-type cocycles.]

Think about homological case: $n=1$



$Y \times S^1$

$\beta_1 \times pt + \beta_2 \times \alpha$.

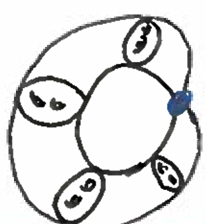
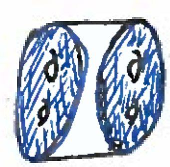
<u>Pf of Lemma:</u> Note that

$$\begin{aligned} H^{n+1}(Y) &\xrightarrow{p^*} H^{n+1}(Y \times S^1) \xrightarrow{i} H^{n+1}(Y) \\ \beta_1 &\longmapsto \beta_1 \times 1 \end{aligned}$$

p^* is 1-1 since the composition:

$$Y \xrightarrow{i} (Y \times \{pt\}) \subseteq Y \times S^1 \xrightarrow{p} Y$$

is the identity. Using the long exact seq for the pair $(Y \times S^1, Y \times \{pt\})$ shows that it is enough to understand

$$Y \times S^1 / Y \times pt =$$  $$\cong Y \times I / Y \times \partial I$$ 

and prove that

$$\begin{aligned} H^n(Y) &\longrightarrow H^{n+1}(Y \times I, Y \times \partial I) \\ \beta &\longmapsto \beta \times \alpha \end{aligned}$$

is an isomorphism, where $\alpha \in H^1(I, \partial I) \cong H^1(S^1)$

Claim ☆