Lecture 4: Today: Cup products and U.C.T for homology.

Cup product:

$$\begin{tikzcd} H^i(X)\times H^j(X) \arrow[r] \arrow[rr, bend right, dashed, "\smile"'] & H^{i+j}(X\times X) \arrow[r, "\Delta^*"] & H^{i+j}(X) \end{tikzcd}$$

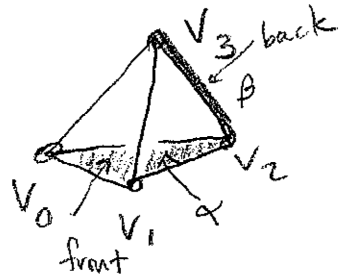
[Intuitively nice but a little messy, see §3.B]

R a ring $\mathbb{Z}$, $\mathbb{Z}/n$, $\mathbb{Q}$, etc. Now define

$$\begin{array}{ccccc} C^k(X;R) & \times & C^l(X;R) & \longrightarrow & C^{k+l}(X;R) \\ \alpha & & \beta & & \alpha\cup\beta \end{array}$$

by

$$(\alpha\cup\beta)(\sigma) = \alpha(\sigma|_{[v_0,\dots,v_k]})\,\beta(\sigma|_{[v_k,\dots,v_n]})$$



$k=2$, $l=1$

where $\sigma:\Delta^{k+l}\to X$ is in $C^{k+1}(X;R)$.

Lemma: $\delta(\alpha\cup\beta) = \delta\alpha\cup\beta + (-1)^k\alpha\cup\delta\beta$

$\sigma:\Delta^{k+l+1}\longrightarrow R$

Pf: $(\delta\alpha\cup\beta)(\sigma) = \underbrace{\delta\alpha}_{\alpha\circ\partial}(\sigma|_{[v_0,\dots,v_{k+1}]})\,\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$

$$\left[\sum_{i=0}^{k+1}(-1)^i\alpha(\sigma_{[v_0,\dots,\hat{v}_i,\dots,v_{k+1}]})\right]\beta(\sigma|_{[v_{k+1},\dots,v_{k+l+1}]})$$

$$(-1)^k\alpha\cup\delta\beta = (-1)^k\alpha(\sigma_{[v_0,\dots,v_k]})\left[\sum_{i=k+1}^{k+l+1}(-1)^{(i-k+1)}\beta(\sigma|_{[v_{k+1},\dots,\hat{v}_i,\dots,v_{k+l+1}]})\right]$$

Last term of first cancels w/ first term of second.

Note: • α, β cocycles then so is $\alpha \vee \beta$

• if $\alpha = \delta\varphi$ then

$$\alpha \cup \beta = \delta\varphi \cup \beta = \delta(\varphi \cup \beta)$$

• if $\beta = \delta\varphi$

$$\alpha \cup \beta = \alpha \vee \delta\varphi = (-1)^k \delta(\alpha \cup \beta)$$

$$\Longrightarrow \quad H^k(X;R) \times H^l(X;R) \xrightarrow{\cup} H^{k+l}(X;R)$$

• associative, distributive; follows from chain level.

Point: Set $H^*(X;R) = \bigoplus_n H^n(X;R)$

This is a ring under cup product. [Note: not nec. comm.]

Moreover, if R has a unit 1_R then so does H^*

$$1_{H^*} \in H^0(X;R) \quad \text{defined by} \quad 1_{H^*}(0\text{-simplex}) = 1_R$$

Ex: $\mathbb{C}P^2$ CW cell structure

$$0 \to \overset{4}{\mathbb{Z}} \to \overset{3}{0} \to \overset{2}{\mathbb{Z}} \to \overset{1}{0} \to \overset{0}{\mathbb{Z}} \to 0$$

	H^n	gens
4	$\mathbb{Z}$	β
3	0	
2	$\mathbb{Z}$	α
1	0	
0	$\mathbb{Z}$	1

$1 \wedge \alpha = \alpha \qquad \alpha \wedge \beta = \beta \wedge \alpha = 0 \qquad \beta \wedge \beta = 0$

$\boxed{\alpha \wedge \alpha = \beta} \qquad H^*(\mathbb{C}P^2) \cong \mathbb{Z}[\alpha]/\alpha^3$

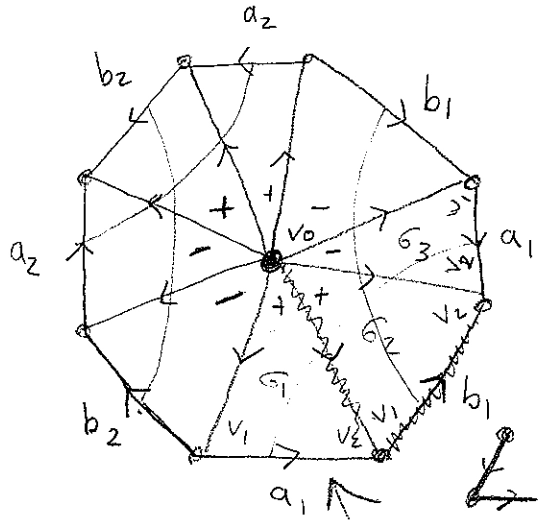
Ex: $S^2 \vee S^4$ same H^*, but $\alpha \wedge \alpha = 0$.

[Hmm... this means its hard to have a cellular def. However there is a simplicial version.]

Ex: $S =$ . Note: order of vertices important.

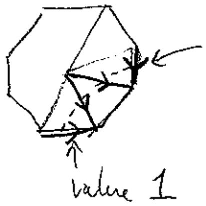
[Must respect axiom of Δ complex: order of verts agree w/ that of each sub, i.e.

]



$$H^1(S;\mathbb{Z}) = \mathrm{Hom}(H_1(S;\mathbb{Z}), \mathbb{Z})$$

$\alpha_1, \beta_1, \alpha_2, \beta_2$ dual basis to a_1, b_1, a_2, b_2.



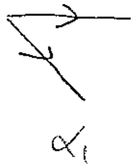
← rep of α_1, say why δ is 0.

$\alpha_1 \wedge \alpha_1(\sigma_1) = \alpha_1(\nearrow)\,\alpha_1(\rightarrow) = 0 \cdot 1 \qquad \alpha_1 \wedge \alpha_1(\sigma_3) = 0$

$\alpha_1 \wedge \alpha_1(\sigma_2) = \alpha_1(\searrow)\,\alpha_1(\nearrow) = 1 \cdot 0 \implies \alpha_1 \wedge \alpha_1 = 0$

a gen [$\alpha_1 \wedge \beta_1(\sigma_2) = 1$

all others 0. $\boxed{\alpha_1 \wedge \beta_1 = \gamma}$



$$H^2(S;\mathbb{Z}) = \mathrm{Hom}(\underset{\substack{\Psi \\ [\gamma]}}{H_2(S;\mathbb{Z})}, \mathbb{Z}) \cong \mathbb{Z}$$

$\implies \alpha_1 \wedge \beta_1 = \gamma$ $[\gamma]$ thus this is gen

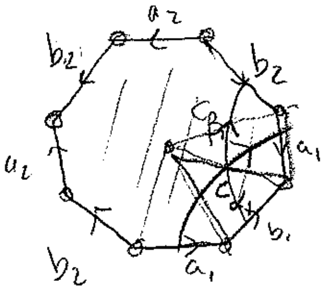
└ given by signs in

$$\beta_1 \wedge \alpha_1(\sigma_3) = 1 \implies \beta_1 \wedge \alpha_1 = -\gamma$$

So

$$\alpha_i \cup \beta_j = \begin{Bmatrix} \gamma & \text{if } i = j \\ 0 & \text{otherwise} \end{Bmatrix} = -\beta_j \wedge \alpha_i$$

Last time:



$$\alpha_i \cup \beta_j = \begin{cases} \gamma & i=j \\ 0 & \end{cases} = -\beta_j \cup \alpha_i$$

all others 0.

N.B. equal to intersection # of $c_{\alpha_i} \cap c_{\beta_i}$

$$\alpha_i \longleftrightarrow c_{\alpha_i}$$

$$H^1 \cong H_1$$

$\cup$ intersection #

Additional properties:

$$\begin{tikzcd}
H^k(X,A)\times H^l(X,A) \arrow[r, "\cup"] & H^{k+l}(X,A) \\
H^k(X,A)\times H^l(X) \arrow[ur, "\cup"] & \\
H^k(\underset{\alpha}{X})\times H^l(\underset{\beta}{X,A}) \arrow[uur, "\cup"] &
\end{tikzcd}$$

Since

$$C^n(X,A) = \mathrm{Hom}\left(C_n(X,A) = C_n(X)/C_n(A), R\right) = \varphi \in C^n(X) \text{ which vanish on } C_n(A).$$

$$\alpha \cup \beta(\sigma \in C_n(A)) = 0$$

Respects induced hom $X \xrightarrow{f} Y$ $\quad H^*(X) \xleftarrow{f^*} H^*(\underset{\alpha,\,\beta}{Y})$

$$f^*(\alpha\cup\beta) = f^*(\alpha) \cup f^*(\beta)$$

[external cup product]

Cross product: $H^k(X)\times H^l(Y) \xrightarrow{\times} H^{k+l}(X\times Y)$

$$\alpha\ ,\ \beta \longmapsto p_1^*(\alpha) \cup p_2^*(\beta)$$

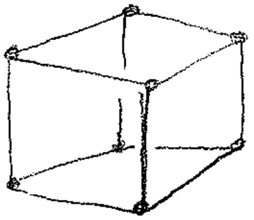
$$\begin{tikzcd}
X\times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \\
X &
\end{tikzcd}$$

$$H^k(X,A)\times H^l(Y,B) \xrightarrow{\times} H^{k+l}(X\times Y, A\times Y \cup X\times B)$$

Ex: $T^n = S^1 \times \cdots \times S^1 = \mathbb{R}^n / \mathbb{Z}^n$.

$$\begin{tikzcd} T^n \arrow[d, "p_i"] \\ S^1 \end{tikzcd}$$

proj onto i^{th} factor



0	1
1	3
2	3
3	1

Fix $\alpha \in H^1(S^1)$ a gen, set $\alpha_i = p_i^*(\alpha)$ then

$H^*(T^n)$ = exterior algebra on $\alpha_1, \dots, \alpha_n$, $\alpha_i \cup \alpha_j = -\alpha_j \cup \alpha_i$

$H^k(T^n)$ = is the free R-module with basis $\alpha_{i_1} \cup \alpha_{i_2} \cup \cdots \cup \alpha_{i_k}$ $i_1 < \cdots < i_k$.

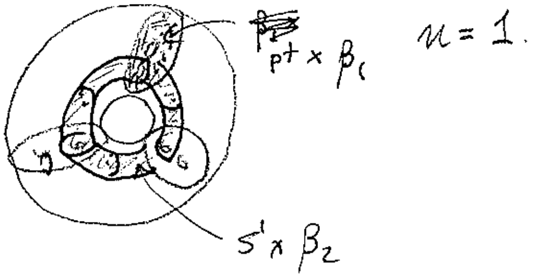
Related to cross product: $n = 4$ $\alpha_1 \cup \alpha_3 = \alpha \times 1 \times \alpha \times 1$
$\alpha_1 \cup \alpha_2 = \alpha \times \alpha \times 1 \times 1$

(*) $H^*(T^n)$ is the free R-module on $\{C_1 \times \cdots \times C_n \mid C_i = 1, \alpha\}$

Lemma: $H^{n+1}(Y) \times H^n(Y) \to H^{n+1}(Y \times S^1)$ is an isomorphism

$$(\beta_1, \beta_2) \longmapsto \beta_1 \times 1 + \beta_2 \times \alpha$$

Think about homology:



[Point: Every cycle decomposes as the sum of two product type cycles.]

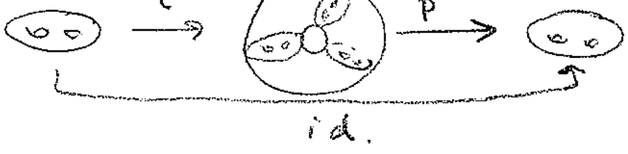
How the lemma applies: prove (*) inductively

Pf of lemma: Note that

$$\begin{tikzcd} H^{n+1}(Y) \arrow[r, "p^*", "proj"'] & H^{n+1}(Y \times S^1) \arrow[r, "i^*", "inc"'] & H^{n+1}(Y) \end{tikzcd}$$

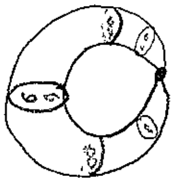
β_1 $\qquad$ $\beta_1 \times 1 = p^*(\beta_1)$

is an isom as



A little messing w/ long exact seq of pair $(Y\times S^1, Y\times\{pt\})$ shows that it is enough to understand

$Y\times S^1 / Y\times pt$



$\cong \quad (Y\times I, Y\times\partial I)$

and show $H^n(Y)\to H^{n+1}(Y\times I, Y\times\partial I)$ is an isom

$\beta \longmapsto \beta\times\alpha \qquad \alpha\in H^1(I,\partial I) \cong H^1(S^1).$

$H^0(\partial I)=\langle 1_0, 1_1\rangle$

$\beta\times(1_0+1_1)$

$\delta|_{\{\beta\times 1_0\}}$ is an isom.

$\beta\times 1_0 \quad \beta\times 1_1$

$$\begin{tikzcd}
 & 0 & H^n(Y)\oplus H^n(Y) & H^n(Y) \arrow[l, "\Delta"'] & 0 \arrow[dl] & \\
\cdots & H^{n+1}(Y\times I, Y\times\partial I) \arrow[l] \arrow[u] & H^n(Y\times\partial I) \arrow[l, "\delta", "\cong"'] & H^n(Y\times I) \arrow[l] & & H^n(Y\times I, Y\times\partial I) \arrow[ll] \arrow[ul] \\
 & H^n(Y)\times H^1(I,\partial I) \arrow[u, "\times"] \arrow[r, phantom, "\circlearrowleft"] & H^n(Y)\times H^0(\partial I) \arrow[l, "\mathrm{id}\times\delta"'] \arrow[u, "\times"', "\cong"] & & &
\end{tikzcd}$$

$1_0, 1_1$

$$\begin{tikzcd}
 & \mathbb{Z}\oplus\mathbb{Z} & \mathbb{Z} \arrow[l] & 0 \\
H^1(I,\partial I) & H^0(\partial I) \arrow[l] & H^0(I) \arrow[l] & H^0(I,\partial I) \arrow[l] \arrow[d, phantom, "\overset{?}{\shortparallel}"] \\
 & \boxed{1_0} + 1_1 \arrow[ul] & \arrow[l] & \tilde{H}(I/\partial I)
\end{tikzcd}$$

$\delta(1_0) = (\text{unit})\alpha$

$$H^n(Y)\times 1_0 \cong H^{n+1}(Y\times I, Y\times\partial I)$$