Lecture 4:

$C_* = \{ \to C_{n+1} \to C_n \xrightarrow{\partial} C_{n-1} \to \}$ chain cpx of free abelian gp

$C^* = \mathrm{Hom}(C_*, G) = \{ \leftarrow C^*_{n+1} \leftarrow C^*_n \xleftarrow[\delta]{} C^*_{n-1} \leftarrow \cdots \}$

$\delta(\varphi) = \varphi \circ \partial$

H_n = homology of C_* $\quad H^n$ = cohomology of C^*

Universal Coeff Thm: The following is split exact:

$$0 \to \mathrm{Ext}(H_{n-1}, G) \to H^n(C_*; G) \xrightarrow{h} \mathrm{Hom}(H_n, G) \to 0$$

Ext: Derived functor of $\mathrm{Hom}(-, G)$.

Free resolution [exact, F_i free] $\quad \to F_2 \to F_1 \to F_0 \to H \to 0$

$$\leftarrow F_2^* \leftarrow F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0$$

$\mathrm{Ext}(H, G)$ = cohomology here.

Last time:
- Explained by Ext is well defined.
- h is onto.

Now let's prove the U.C.T.

$Z_n = \ker \partial_n \quad B_n = \mathrm{im}\, \partial_{n+1}$

Pf of U.C.T. $0 \to \ker h \to H^n(C_*;G) \to \mathrm{Hom}(H_n,G) \to 0$

Short exact seq of chain complexes

$$\begin{tikzcd}
 & \downarrow & \downarrow & \downarrow & \\
0 \arrow[r] & Z_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "0"'] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "i_n"] \arrow[d] & C_n \arrow[r, "\partial"] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & 0 \\
 & & & &
\end{tikzcd}$$

Each row splits, since terms are free

$\Downarrow$

Each row is exact.

Apply $\mathrm{Hom}(\_\,,G)$:

$\varphi_1\big|_{B_n} = \varphi_0\big|_{B_n}$

$$\begin{tikzcd}
 & \uparrow & \uparrow & \uparrow & \\
0 & Z^*_{n+1} \arrow[l] & C^*_{n+1} \arrow[l, "\varphi_1\circ\partial"'] & B^*_n \arrow[l, "\delta"'] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l, "i_n^*"'] \arrow[u, "\delta"] & B^*_{n-1} \arrow[l, "\delta"'] \arrow[u, "0"'] & 0 \arrow[l] \\
 & \uparrow & \uparrow & \uparrow &
\end{tikzcd}$$

$\varphi_0 \longleftrightarrow \varphi_1$

Take the long exact seq.

$$\leftarrow B^*_n \xleftarrow{i_n^*} Z^*_n \leftarrow H^n(C,G) \xleftarrow{\delta} B^*_{n-1} \xleftarrow{i_{n-1}^*} Z^*_{n-1} \leftarrow$$

The connecting homomorphism is just the dual to the inclusion $B_n \xrightarrow[i_n]{} Z_n$.

Hence

$$\begin{tikzcd} 0 & \ker i_n^* \arrow[l] \arrow[d, equal] & H^n(C,G) \arrow[l, "h"'] & \operatorname{coker} i_{n-1} \arrow[l] \arrow[d, equal] & 0 \arrow[l] \\ & \mathrm{Hom}(H_n, G) & & \mathrm{Ext}(H_{n-1}, G) & \end{tikzcd}$$

<u>Point</u>: $0 \to B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \to H_{n-1} \to 0$ Free resolution

$$0 \leftarrow B_{n-1} \xleftarrow{i^*_{n-1}} Z^*_{n-1} \leftarrow H^*_{n-1} \leftarrow 0$$

Cohomology here is $\mathrm{Ext}(H_{n-1}, G)$ ∎

<u>Universal Coeff Thm for Homology</u>: C_* chain complex of free abelian gps. The following is split exact:

$$0 \to H_n(C) \otimes G \to H_n(C ; G) \to \mathrm{Tor}(H_{n-1}(C), G) \to 0$$

[Query: Familiar with $\otimes$?]

$$\cdots \to F_2 \to F_1 \to F_0 \to A \to 0 \quad \text{Free resolution}$$

$$\to F_2 \otimes G \to F_1 \otimes G \to F_0 \otimes G \to A \otimes G \to 0$$

Homology here is $\mathrm{Tor}(A, G)$

<u>Props:</u> $\mathrm{Tor}(A,B) \cong \mathrm{Tor}(B,A)$

$\mathrm{Tor}(\oplus A_i, B) \cong \oplus_i \mathrm{Tor}(A_i, G)$

$\mathrm{Tor}(A,B) = 0$ if A is torsion free

$\mathrm{Tor}(\mathbb{Z}/n, B) = \mathrm{Ker}(B \xrightarrow{n} B)$

Suppose $H_n(C) = \mathbb{Z}^d \oplus \mathbb{Z}/p^{k_1} \oplus \mathbb{Z}/p^{k_2} \cdots \oplus \mathbb{Z}/p^{k_n} \oplus$ torsion coprime to p

$H_{n-1}(C) = \mathbb{Z}^e \oplus \mathbb{Z}/p^{l_1} \oplus \cdots \oplus \mathbb{Z}/p^{l_m} \oplus$ torsion coprime to p

<u>Ex:</u> $H_n(C;\mathbb{Z}/p) \cong (\mathbb{Z}/p)^{d+n+m} \cong H^n(C;\mathbb{Z}/p)$

Compare $H_*(\mathbb{R}P^n)$. [Work one prime at a time.]

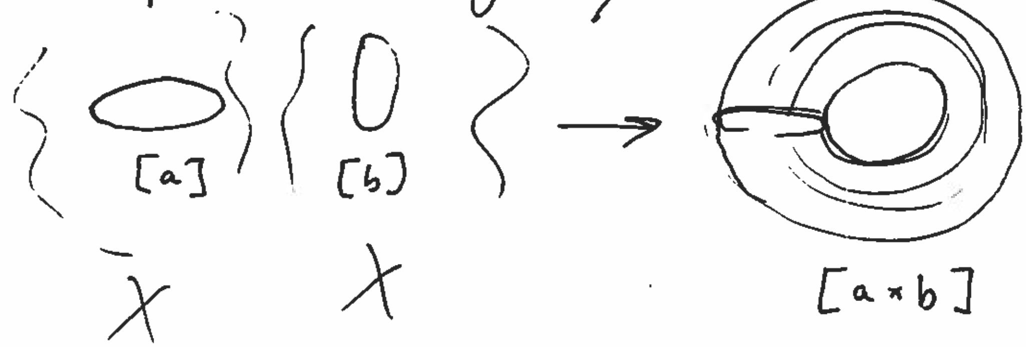
<u>Ex:</u> $H_n(C;\mathbb{Q}) \cong \mathbb{Q}^d \cong H^n(C;\mathbb{Q})$

<u>Cup product:</u> X CW complex

Look at H^* with $G = R$ a ring $[\mathbb{Z}, \mathbb{Q}, \mathbb{Z}/n, \ldots]$

$$H^k(X) \times H^l(X) \longrightarrow H^{k+l}(X \times X)$$

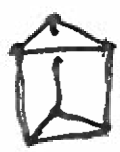
Think first homologically



$X \times X$ — $\underbrace{X \times X}$ cell str with product of cells

On cochains $\varphi \in C^k(X)$ + $\psi \in C^\ell(X)$ define

$$(\varphi \times \psi)(\underbrace{\sigma \times \tau}_{\text{product cell}}) = \begin{cases} \varphi(\sigma)\cdot\psi(\tau) & \text{if } \dim \sigma = k, \ \dim \psi = \ell \\ 0 & \text{otherwise.} \end{cases}$$



Gives map on cohomology.

Consider $\Delta : X \longrightarrow X \times X$

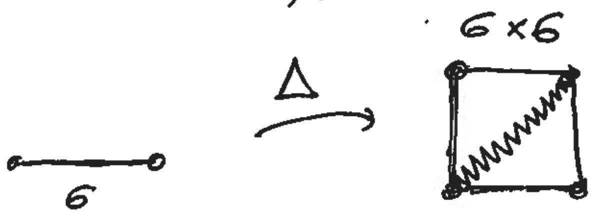
$x \longmapsto (x, x)$

Cup product: $[\varphi] \cup [\psi] = \Delta^*[\varphi \times \psi]$

$$\begin{array}{ccccc} H^k(X) \times H^\ell(X) & \longrightarrow & H^{k+\ell}(X \times X) & \xrightarrow{\Delta} & H^{k+\ell}(X) \\ [\varphi] \quad [\psi] & \longrightarrow & [\varphi \times \psi] & \longrightarrow & \Delta^*[\varphi \times \psi] \end{array}$$

<u>Techical Issue</u>: Δ is not cellular

X: $\quad X\times X$:



Work arounds: See §3.B

- homotope Δ to a cellular map
- Define $\times$ on singular homology

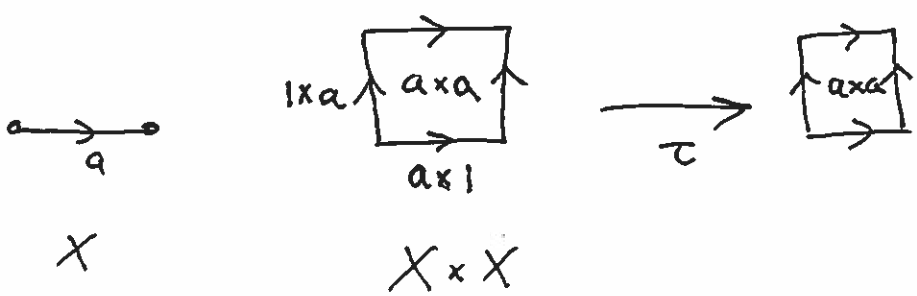
Will actually define another way on Friday.

The cup product will make $H^*(X;R) = \bigoplus_k H^k(X;R)$ into a "ring" which is associative and distributive but not quite commutative: $\alpha \cup \beta = (-1)^{kl} \beta \cup \alpha$.

Reason

$\tau : X \times X \circlearrowleft \qquad \tau(x_1, x_2) = (x_2, x_1)$

does not act trivially on $H^*(X,X)$. In odd dimensions have



$\tau_*(a\times a) = -a\times a$