Lecture 15: First proof of Poincaré Duality

M an n-mfld with simplical triangulation $\mathcal{T}$

(so $\mathcal{T} = \Delta^n$'s with $n-1$ faces glued in pairs)

$$\sigma \text{ in } \mathcal{T} \longrightarrow \begin{cases} D(\sigma) = \bigcup\{\text{int}(\alpha) \mid \alpha \text{ in } sd(\sigma) \text{ has last vertex } \hat{\sigma}\} \\ \bar{D}(\sigma) = \bigcup\{\alpha \mid \text{same}\} \\ \dot{D}(\sigma) = \bar{D}(\sigma) - D(\sigma) \end{cases}$$

[Draw 2-d and 3-d pictures]

(a) $D(\sigma)$ are disjoint with union M

(b) $\bar{D}(\sigma)$ is a subcomplex of $sd(\mathcal{T})$ of dim $n - |\sigma|$

(c) $\dot{D}(\sigma) = \{D(\tau) \mid \sigma \not\leq \tau\}$

$\mathcal{D}$ = "cell" complex of the $D(\sigma)$.

Def: $\mathcal{T}$ is PL if every $\bar{D}(\sigma)$ is homeo to a ball $D^{n-|\sigma|}$.

[I will assume this but its not actually needed; independent of the topology of the $\bar{D}(\sigma)$, homologically they are balls; also any smooth mfld has one...]

Thm: M cpt connected n-mfld with a PL triangulation $\mathcal{T}$.

Then $H^k(M;\mathbb{F}_2) \cong H_{n-k}(M;\mathbb{F}_2)$

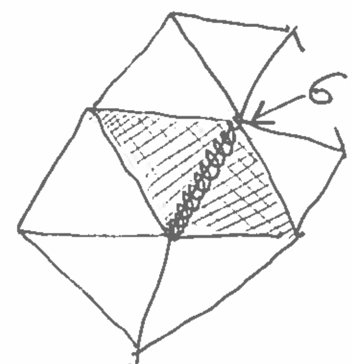
Proof:

cochains of $\mathcal{T}$

$$
\begin{tikzcd}
C^{k+1} \arrow[r, "\cong"] & D_{n-k-1} \\
C^{k} \arrow[u, "\delta"] \arrow[r, "\cong"'] & D_{n-k} \arrow[u, "\partial"']
\end{tikzcd}
$$

chains of $\mathcal{D}$

Can think of D_{n-k} as a union of $n-k$ cells of $\mathcal{D}$ (as coeffs in $\mathbb{F}_2$). Same with C^k as unions of k cells of $\mathcal{T}$. So for σ_k in $\mathcal{T}$ have both $\sigma_k^* \in C^k$ and $D(\sigma_k)$ in D_{n-k}. and hence horizontal isomorphisms. It remains to show that the diagram commutes. First,

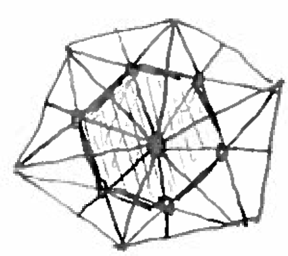


$$\delta\sigma^* = \bigcup\left\{\tau \;\middle|\; \begin{matrix} |\tau| = |\sigma| + 1 \\ \tau > \sigma \end{matrix}\right\}$$

Second, topologically the boundary of $D(\sigma)$ is $\dot{D}(\sigma) = \bigcup\{D(\tau) \mid \tau \gneq \sigma\}$. Hence

$$\partial D(\sigma) = \bigcup\{D(\tau) \mid \tau > \sigma \text{ and } |\tau| = |\sigma| + 1\}$$

So the diagram commutes!



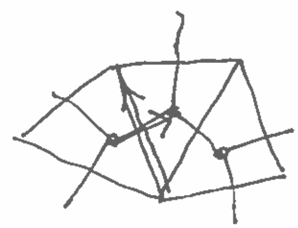
Poincaré Duality holds! ∎

Same proof ("turn cell decomp upside down") works for $\mathbb{Z}$ if we orient things carefully.
[There's an annoying inductive way to do this...]

<u>Cap product on homology</u>: Continue with $\mathbb{F}_2$ coeffs.

$$\begin{array}{ccc} H_k(M) \times H_{n-k}(M) & \longrightarrow & \mathbb{F}_2 \\ \alpha \quad\quad\quad \beta & & \alpha \cap \beta \end{array}$$

Where for $c \in C_k$ and $d \in D_{n-k}$ we have



$$\alpha \cap \beta = \#(c \cap d) \mod 2.$$

[This turns out to be ✓ well-defined and makes sense over $\mathbb{Z}/R$]

<u>Thm</u>: M closed connected with PL triangulation $\mathcal{T}$. Then $\cap$ is a nondegen. bilinear form on $H_k(M) \times H_{n-k}(M)$

<u>Pf</u>: Pick $[\varphi]$ in $H^k(M)$ with $\varphi \in C^k$ and $\varphi(c) = 1$. Then

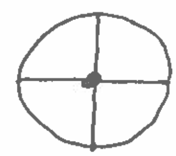
$$c \cap \underbrace{D(\varphi)}_{\text{in } H_{n-k}} = \varphi(c) = 1. \qquad ▨$$

Also, if η and ψ are the Poincaré duals of α and β, then you can check that

$$\alpha \cap \beta = (\eta \cup \psi)[M]$$ ($\Rightarrow$ Invariance of $\cap$ on homology.)

[For general M, once have P.D. can use to define cup prod.]

Now easy to see that $H^*(\mathbb{RP}^n; \mathbb{F}_2) = \mathbb{F}_2[\alpha] / \alpha^{n+1} = 0$ $\quad |\alpha| = 1$.



Types of manifolds:

<u>TOP</u>: Topological manifolds and cont maps.

$\uparrow$

<u>PL</u>: Mflds with PL-triangulations and PL-maps.

$\uparrow$

<u>DIFF</u>: Smooth mflds and maps.

In general, neither are injective or surjective...

<u>Next</u>: How to prove Poincaré Duality inducively.