Lecture 13: Poincaré Duality 101

Lemma: $A^{cpt} \subseteq M^n$ Then

(a) $H_k(M|A) = 0$ for $k > n$.

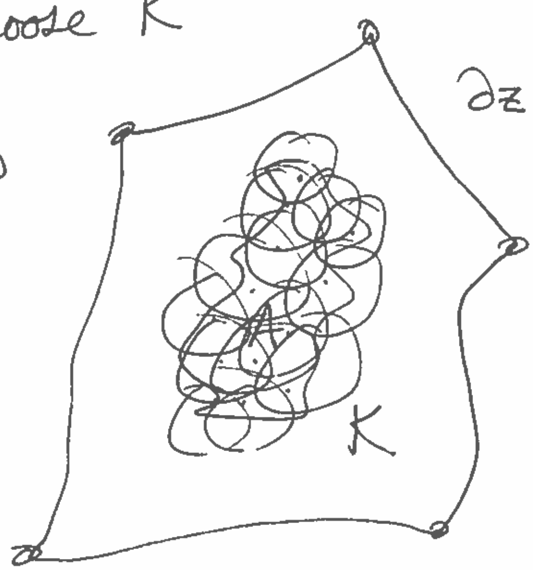
(b) Given a section $x \longmapsto \alpha_x$ of $M_{\mathbb{Z}} \to M$ $\exists!\ \alpha_A \in H_n(M|A)$ which maps to α_x in each $H_n(M|x)$ for $x \in A$.

Final bit of Pf: Uniqueness of α_A when $A^{cpt} \subseteq \mathbb{R}^n$.

Suppose α_A and α'_A are two such. Then

$$\alpha_A - \alpha'_A \longmapsto 0 \text{ in each } H_n(M|x) \text{ for } x \in A.$$

Must show $\alpha_A - \alpha'_A = 0$ in $H_n(M|A)$. Let z be a cycle rep $\alpha_A - \alpha'_A$ and choose K as last time. Then $[z] \to 0$ in $H_n(M|x)$ for all x in K



Reason: Know for $x \in A$ and

both $H_n(M|x)$ and $H_n(M|y)$ are isom to $H_n(M|B)$

under inclusion.

So $[z] \in H_n(M|K)$ is an α_{K}, 0-section,

and so by uniqueness $[z] = 0$ in $H_n(M|K)$ and

hence in $H_n(M|A)$ as needed. $\blacksquare$

Note for any ring R can consider

$$M_R = \{ \alpha_x \in H_n(M|x) \mid x \in M \}$$

as a cover of M. [Have notion of R-orientability; same lemma, etc.] Any mfld is $\mathbb{F}_2$ orientable, for instance.

Poincaré Duality: M^n closed conn. orientable

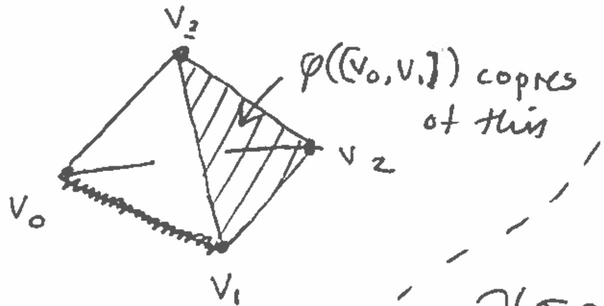
$$H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$$

[How this is mediated; consequences and interpretations.]

<u>Cap product:</u> X space, R = coeff ring

$$\cap : \underset{\substack{\in \\ \sigma : \Delta^k \to X}}{C_k(X)} \times \underset{\substack{\in \\ \varphi}}{C^\ell(X)} \longrightarrow C_{k-\ell}(X) \qquad \boxed{k \geq \ell}$$

$$\sigma \cap \varphi = \varphi(\sigma|_{[v_0, \dots, v_\ell]})\, \sigma|_{[v_\ell, \dots, v_k]}$$



$k=3,\ \ell=1$

You can check that

$$\partial(\sigma \cap \varphi) = (-1)^\ell(\partial\sigma \cap \varphi - \sigma \cap \delta\varphi)$$

So get an R-bilinear map

$$H_k(X) \times H^\ell(X) \longrightarrow H_{k-\ell}(X)$$

<u>Naturality:</u> $X \xrightarrow{f} Y \quad \alpha \in H_k(X) \quad \varphi \in H^\ell(Y)$

Then $f_*(\alpha) \cap \varphi = f_*(\alpha \cap f^*(\varphi))$

<u>Poincare Duality</u>: M (closed conn) is R-orientable, $[M] \in H_n(M;R)$ a generator. Then $D : H^k(M) \longrightarrow H_{n-k}(M)$ given by $D(\alpha) = [M] \cap \alpha$ is an isomorphism.

<u>Cor.</u> M^n closed conn. Then $H_k(M;\mathbb{F}_2) \cong H_{n-k}(M;\mathbb{F}_2)$ since $H^k(M;\mathbb{F}_2) \cong H_k(M;\mathbb{F}_2)$.

F is a field. $C^k(X;F) = \mathrm{Hom}(C_k(X;\mathbb{Z}), F)$

X finite cell complex

$$= \mathrm{Hom}_F(C_k(X;F), F)$$
$$= C_k(X;F)^*$$

For the same reason as in proof of U.C.T. get that since $V^{**} \cong V$.

$$\begin{cases} H^k(X;F) \longrightarrow H_k(X;F)^* \longrightarrow 0 \\ H_k(X;F) \longrightarrow H^k(X;F)^* \longrightarrow 0 \end{cases}$$

<u>Cor.</u> The Euler characteristic of any odd dim'l closed mfld is 0.

Pf: $\chi(M) = \sum (-1)^i \operatorname{rank}(H_i(M;\mathbb{Z}))$

$$\overset{\star}{=} \sum (-1)^i \dim(H_i(M;\mathbb{F}_2))$$

As n is odd, i and $n-i$ have opposite parity

Hence by corollary these terms cancel in pairs.

Reason for $\star$.

(a) If M is a finite cell complex, then both are $= \sum (-1)^i (\#\text{ of } i \text{ cells})$

(b) If M has finitely gen $H_*(M;\mathbb{Z})$ then follows from U.C.T. since

$$\dim H^i(M;\mathbb{F}_2) = \operatorname{rank} H_i(M;\mathbb{Z}) + \operatorname{rank} \begin{matrix}\text{2-part of}\\ H_i(M;\mathbb{Z})\end{matrix} + \operatorname{rank} \begin{matrix}\text{2-part of}\\ H_{i-1}(M;\mathbb{Z})\end{matrix}$$

In fact any closed n-mfld is homotopy equivalent to a CW complex

[Hatcher A.12]