Lecture 12: Orientations and homology II.

M an n-mfld. Define $M_{\mathbb{Z}} \to M$ by

$$M_{\mathbb{Z}} = \{\alpha_x \in H_n(M|x;\mathbb{Z}) \mid x \in M\}$$

Lemma: $A \subseteq M$ cpt. Then

ⓐ $H_i(M;\mathbb{Z}) = 0$ for $i > n$.

ⓑ If $x \mapsto \alpha_x$ is a section of $M_{\mathbb{Z}} \to M$ then $\exists!$ $\alpha_A \in H_n(M|A;\mathbb{Z})$ whose image in $H_n(M|x;\mathbb{Z})$ is α_x for all $x \in A$.

Thm: M clsd conn. n-mfld. Then

- $H_i(M;\mathbb{Z}) = 0$ for $i > n$.
- $H_n(M;\mathbb{Z}) = \begin{cases} \mathbb{Z} & M \text{ orientable} \\ 0 & \text{otherwise} \end{cases}$

Pf of lemma: Main steps:

① If lemma holds for A, B and $A \cap B$, then holds for $A \cup B$.

② Can assume $M = \mathbb{R}^n$.

③ Holds for finite unions of cpt convex $\subseteq \mathbb{R}^n$.

④ Holds for all cpt $\subseteq \mathbb{R}^n$.

(1) Consider the Mayer–Vietoris sequence assoc to

$$X \cap Y \to X \amalg Y \to X \cup Y$$

where $X = (M, M \setminus A)$ and $Y = (M, M \setminus B)$, that is

$$0 \to H_n(M \mid A \cup B) \xrightarrow{\Phi} H_n(M \mid A) \oplus H_n(M \mid B) \xrightarrow{\Psi} H_n(M \mid A \cap B)$$

since $A \cap B$ sat the lemma.

$$\alpha \longmapsto (\alpha, \alpha) \qquad (\alpha, \beta) \longmapsto \alpha - \beta$$

Conclusion (a) immediately holds since $0 \to H_n(M \mid A \cup B) \to 0 \oplus 0$

Let α_A, α_B and $\alpha_{A \cap B}$ be the unique classes given in (b). Note that both α_A and α_B map to $\alpha_{A \cap B}$ under inclusion, and hence $\Psi(\alpha_A, \alpha_B) = 0$.

Define $\alpha_{A \cup B}$ by $\Phi(\alpha_{A \cup B}) = (\alpha_A, \alpha_B)$.

For all $x \in A$, have $\alpha_{A \cup B} \mapsto \alpha_A \mapsto \alpha_x \in H_n(M \mid x)$ and sim for $x \in B$. So $\alpha_{A \cup B}$ sat the condition of (b). Note that $\alpha_{A \cup B}$ is unique as as otherwise would get alternate choice for at least one of α_A and α_B.

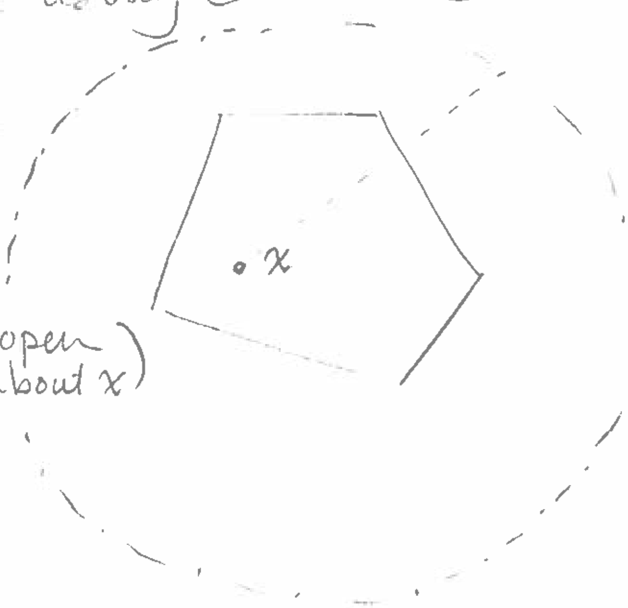
2 Any cpt $A \subseteq M$ is $\bigcup_{i=1}^{m} A_i$ where each A_i is cpt and contained in an open $U_i \cong \mathbb{R}^n$.

Assuming (a) and (b) hold for cpt in $\mathbb{R}^n$, induct on m to show it holds for such finite unions of $(\text{cpt} \subseteq \mathbb{R}^n \subseteq_{\text{open}} M)$. That is, apply 1 to $(A_1 \cup \cdots \cup A_{m-1})$ and A_m, noting that their intersection is $(A_1 \cap A_m) \cup \cdots \cup (A_{m-1} \cap A_m)$ which we know is OK by induction.

3 As intersections of convex sets are convex, it is enough to prove (a)+(b) for convex sets; finite unions are handled inductively using 1 as in 2.

If A is convex, consider $x \in A$ then both $\mathbb{R}^n \setminus \{x\}$ and $\mathbb{R}^n \setminus A$ deformation retract to $\mathbb{R}^n \setminus$ (large open ball about x)

ball about.



In particular $H_n(\mathbb{R}^n | A) \longrightarrow H_n(\mathbb{R}^n | x)$ is an isom. So define $\alpha_A = i_*^{-1}(\alpha_x)$ for some $x \in A$.

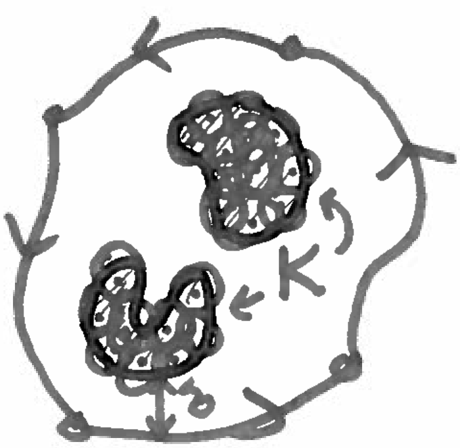
Continuity of the section $x \longmapsto \alpha_x$ means that α_A works for all other $y \in A$.

[Where did I use convexity of A?]

4. Let $A \underset{\text{cpt}}{\subseteq} \mathbb{R}^n$. Suppose $\alpha \in H_i(\mathbb{R}^n | A)$ is represented by z. Let $C = \bigcup_{\text{simplices in } \partial z}$ images of $\subseteq \mathbb{R}^n \setminus A$

Let $\delta = \text{dist}(A, C) > 0$, and cover A by finitely closed balls of radius $\delta/2$; call

$z = \sigma_1 + \sigma_2$



their union K.

Note z is also a relative cycle for $\mathbb{R}^n \setminus K$, giving α_K in $H_i(\mathbb{R}^n|K)$ mapping to the original $\alpha \in H_i(\mathbb{R}^n|A)$. If $i > n$, then $\alpha_K = 0$ and so $H_i(\mathbb{R}^n|A) = 0$ proving (a). For (b), existence is easy: just take α_B where B is some large compact ball containing A. For uniqueness, suppose z reps $\alpha \in H_n(\mathbb{R}^n|A)$ so that $\alpha \mapsto 0$ in $H_n(\mathbb{R}^n|x)$ for all $x \in A$. <u>Claim</u>: $\alpha = 0$.

Let z as above rep α. As K is a union of balls and $H_n(\mathbb{R}^n|B) \longrightarrow H_n(\mathbb{R}^n|x)$ for all $x \in B$, it follows that $\alpha_K \longmapsto 0$ for all $x \in K$. By (3), the uniqueness claim of (b) implies that $\alpha_K = 0$ in $H_n(\mathbb{R}^n|K)$ and hence $\alpha = 0$ as claimed. ∎