Lecture 10: Homology of Manifolds

Def: An n-manifold is a Hausdorff, 2^{nd} countable, topological space where every pt has an open nbhd homeo to $\mathbb{R}^n$.

[Geometric topology: study of such. For now, H_* and H^*]

= cpt and w/o boundary

Poincaré Duality: M a closed connected n-mfld. Then $H_k(M;\mathbb{F}_2) \cong H_{n-k}(M;\mathbb{F}_2)$. If M is orientable then $H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z})$.

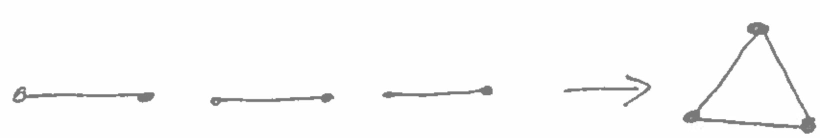
[Surprising since being a mfld is a purely local cond.]

Thm. M closed conn n-mfld. Then $H_n(M;\mathbb{Z}) = \mathbb{Z}$ or 0 and $H_n(M;\mathbb{F}_2) = \mathbb{F}_2$.

orientable non-orient.

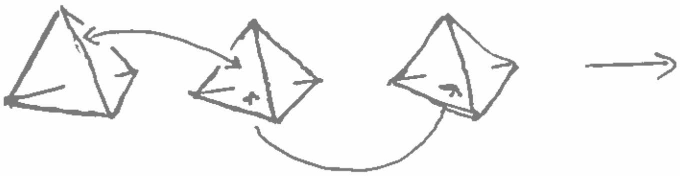
Def. A triangulation of M is a Δ-complex str consisting of n-simplices w/ their $n-1$ faces glued in pairs.

Ex: $n=1$



<u>$n = 2$:</u>

<u>$n = 3$:</u>

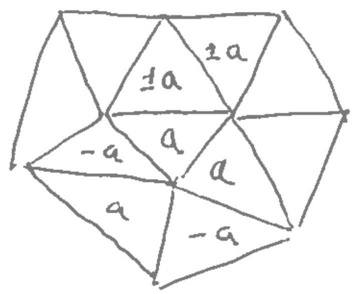


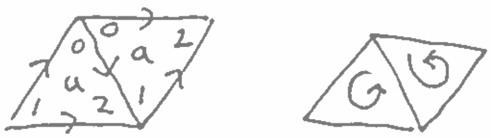
M^3

Suppose M has a triangulation: Then $H_n(M;\mathbb{F}_2) = \mathbb{F}_2$

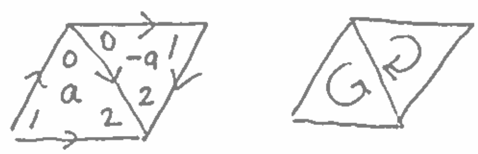
[Q: What is the gen? Q: Why conn? Q: cpt?]

What about $H_n(M;\mathbb{Z})$?





vs.



Either everything fits together: $H_n(M;\mathbb{Z}) = \mathbb{Z}$

or not $\quad H_n(M;\mathbb{Z}) = 0$

Möbius band

Q: Does every cpt n-mfld have a triangulation?

A: No! [Manolescu 2013] [Every smooth one does, though.]

Orientation of $\mathbb{R}^n$: [preserved under rotations, switch under reflections]

An orient of $\mathbb{R}^n$ at x is a choice of generator in

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \underset{\uparrow}{\cong} H_{n-1}(\mathbb{R}^n \setminus \{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

long exact seq.

Suppose B open ball with $x \in B$.

$$\begin{tikzcd} & H_n\left(S^n = \mathbb{R}^n / \mathbb{R}^n \setminus B\right) \arrow[d, equal] \\ H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \arrow[l, "\cong"', "i_*"] \arrow[d, "\cong"', "i_*"] \\ & H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{y\}) \end{tikzcd}$$

So orient here determines one here

Local homology of X at A:

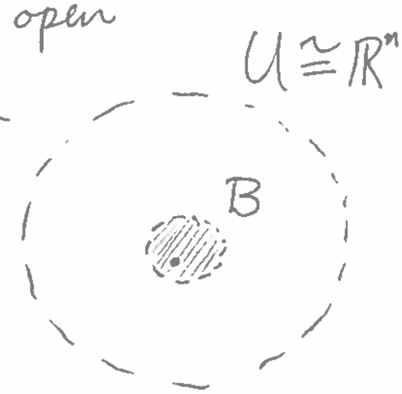
$$H_n(X \mid A) = H_n(X, X \setminus A)$$

If M is an n-mfld $\quad H_n(M \mid x) \cong H_n(\mathbb{R}^n \mid pt)$

[Q: Excision]

A local orient at $x \in M$ is a choice of gen μ_x of $H_n(M \mid x) \cong \mathbb{Z}$.

Def: An *orientation* of M is a fn $x \longmapsto \mu_x$ such that $\forall$ open sets $U \cong \mathbb{R}^n$ and bounded open balls $B \subset U$ there exists $\mu \in H_n(M|B)$ such that



$$\begin{array}{ccccc} H_n(M|x) & \longleftarrow & H_n(M|B) \cong H_n(U|B) \\ \mu_x & \longleftarrow\!\shortmid & \mu \qquad\qquad \cong \mathbb{Z} \end{array}$$

for all $x \in B$.

If an orient. exists, M is called *orientable*.

Thm: M clsd connected n-mfld. If M is orient. then $H_n(M;\mathbb{Z}) = \mathbb{Z}$ and $H_n(M;\mathbb{Z}) \longrightarrow H_n(M|x;\mathbb{Z})$ is an isomorphism for all $x \in M$. Otherwise, $H_n(M;\mathbb{Z}) = 0$.

Note: ① Easy to see that ↑ implies orientability.

Fix a gen μ of $H_n(M;\mathbb{Z})$ and set μ_x = image in $H_n(M|x;\mathbb{Z})$.

② Any mfld is $\mathbb{F}_2$-orientable, since $H_n(M|x;\mathbb{F}_2) \cong \mathbb{F}_2$ has a single non-zero element.