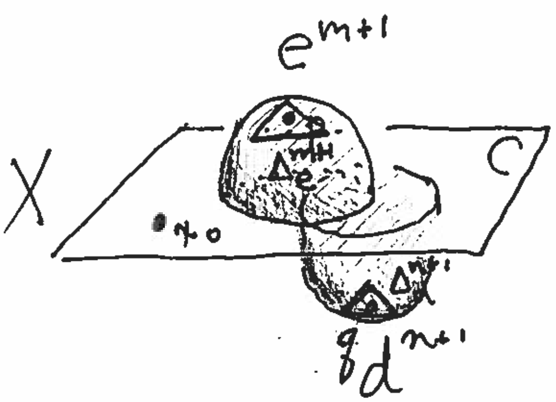
Lecture 26: Rest of proof of excision

Excision: X a CW complex which is the union of subcomplexes A and B with $C = A \cap B \neq \emptyset$. In (A, C) is m-conn and (B, C) is n-connected, then $\pi_i(A, C) \to \pi_{i+1}(X, B)$ is an isom for $i < m+n$ and onto for $i = m+n$.

Lemma: $f: I^i \to (W \cup e^k)$. Then f is homotopic, rel $f^{-1}(W)$ to a map f_1 such that $\exists$ a simplex $\Delta^k \subseteq \text{int}(e^k)$ where $f_1^{-1}(\Delta^k)$ is a finite union of convex polyhedra on which f_1 is the restriction of a projection $\mathbb{R}^i \to \mathbb{R}^k$. [Lemma 4.10 in Hutcher.]

[Query: What does lemma mean if $i < k$?]

Pf of Excision: $A = C \cup e^{m+1}$ $\quad B = C \cup d^{n+1}$



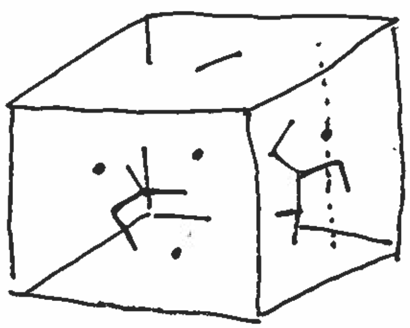
$$\begin{array}{ccc} \pi_i(A, C) & \longrightarrow & \pi_i(X, B) \\ \cong \downarrow & & \downarrow \cong \\ \pi_i(X \setminus \{q\}, X \setminus \{p, q\}) & \longrightarrow & \pi_i(X, X \setminus \{p\}) \end{array}$$

Trying to show onto. $f: (I^i, \partial I, J) \to (X, B, x_0)$

Apply Lemma. $f^{-1}(p) =$ polyhedra of dim $i - (m+1)$

$f^{-1}(q) =$ polyhedra of dim $i - (n+1)$

Perturb p so no point of $f^{-1}(p)$ is on the same vertical line as a pt in $f^{-1}(q)$. Can



do when $m + n \le i$ since

$\dim\left(f\left(\underbrace{f^{-1}(q) \times I}_{\text{explain}}\right) \cap \Delta_e^{m+1}\right)$

is $\le i - n$ and hence $< \dim \Delta_e^{m+1}$.

$m = 1$

$n = 2$

Now argue as before.

For 1-1 when $m + n < i$, basically the same argument but are trying to push back a homotopy, i.e. a map $I^{i+1} \to (X, B)$, so lose a dimension.

Next case: $A = C \cup {}^{\text{many}}_{m+1 \text{ cells}}$

$B = C \cup {}^{\text{many}}_{n+1 \text{ cells}}$

When A or B has cells above dim $m+1 / n+1$ induct on skelata.

Cor Prop 4.13: X is homotopy equivalent, fixing C, to X' where $A' = C \cup$ cells of dim $> m$

$B' = C \cup$ cells of dim $> n$. ▨

Eilenberg-MacLane spaces: If $\pi_n(X) = G$ and all other $\pi_i(X) = 0$, call X a $K(G, n)$.

Ex: S^1 is a $K(\mathbb{Z}, 1)$ $\quad T^n = (S^1)^n$ is a $K(\mathbb{Z}^n, 1)$

In general, a CW complex X is a $K(G,1)$ iff $\pi_1 X = G$ and $\tilde{X}$ is contractible.

Ex: X a graph $\qquad X =$ closed orient surface of genus > 0.

Ex: $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z}, 1)$ [What's the universal cover? S^∞]

Ex: $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ [It turns out.]

$\pi_1 = 1$

$\pi_2 = \pi_2((\mathbb{C}P^\infty)^{(3)}) = \pi_2(S^2) = \mathbb{Z}$.

Thm: For any group G, there is a $K(G,1)$.
If G is abelian, there is a $K(G,n)$ for all $n \geq 1$.
These spaces are unique up to homotopy equivalence.

[Will give the proof next lecture.]

Def: X, Y spaces with base points x_0, y_0

$\langle X, Y \rangle =$ base pt pres. homotopy classes of maps $(X, x_0) \to (Y, y_0)$.

Ex: $\pi_n X = \langle S^n, X \rangle$

Thm: Let X be a CW complex, G an abelian gp, $n > 0$. Then $\exists$ a natural bijection

$$T : \langle X, K(G,n) \rangle \longrightarrow H^n(X; G)$$

which has the form $T([f]) = f^*(\alpha)$ for a certain fixed $\alpha \in H^n(K(G,n); G)$.

<u>Ex:</u> $G = \mathbb{Z}$, $n = 1$ $\qquad H^1(X;\mathbb{Z}) \cong \langle X, S^1 \rangle$

S^1 is a $K(G,n)$ $\qquad f^*([S^1]^*) \longleftarrow (X \xrightarrow{f} S^1)$

generator of $H^1(S^1;\mathbb{Z})$
which evals to one on $[S^1]$.

<u>One way to think about:</u> Suppose X is connected.

Then $H^1(X;\mathbb{Z}) = \mathrm{Hom}(H_1(X), \mathbb{Z}) = \mathrm{Hom}(\pi_1 X, \mathbb{Z})$

Suppose $X \xrightarrow{f} S^1$. Get $\pi_1 X \xrightarrow[f_*]{} \pi_1 S^1 = \mathbb{Z}$

The isom above is just $f \longmapsto f_* \in \mathrm{Hom}(\pi_1 X, \mathbb{Z})$

since

$$f^*([S^1]^*)(\alpha) = [S^1]^*(f_*\alpha) = f_*(\alpha)$$

$\alpha \in \pi_1 X / H_1(X)$ $\qquad$ under the ident of $\pi_1 S^1$ with $\mathbb{Z}$.

Thus the Thm is equivalent to

- Any $\phi : \pi_1 X \to \mathbb{Z}$ can be realized by some $f : X \to S^1$
- Any two such realizations are homotopic.

Not hard to see geometrically...