Lecture 22: More on Relative Homotopy Groups.

$\pi_n(X, A, x_0) =$ homotopy classes of maps $(D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$

$(I^n, I^{n-1}, J^{n-1}) \longrightarrow (X, A, x_0)$, where $J^{n-1} = I^n \setminus \mathrm{Int}(I^{n-1})$

Long exact sequence:

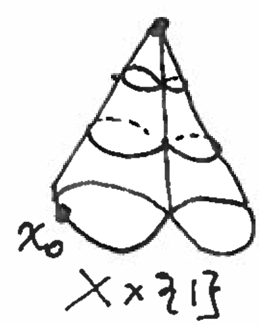
$$\to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \pi_{n-1}(X, x_0) \to \pi_{n-1}(X, A, x_0) \to \cdots \to \pi_1(X, A, x_0) \to \pi_0(A, x_0) \to \pi_0(X, x_0)$$

Compression Criterion: $f : (D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$ is 0 in $\pi_n(X, A, x_0)$ iff it is homotopic, rel S^{n-1}, to a map with image in A.

Ex: $CX = X \times [0,1] \Big/ \substack{\text{collaspe} \\ X \times \{0\}}$

$\cup|$

X as $X \times \{1\}$



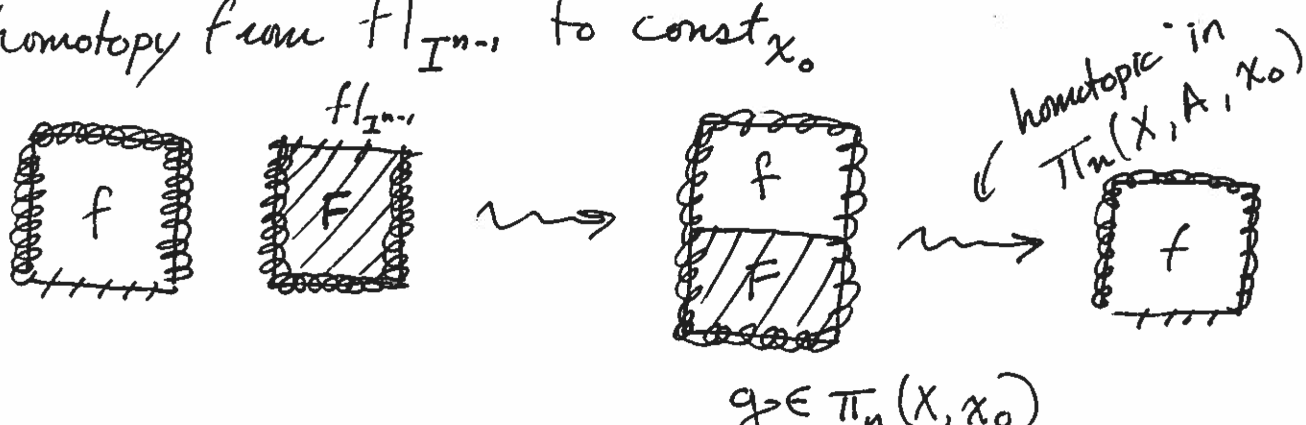
$\boxed{\text{So can make relative } \pi_2 \text{ anything}}$

$$\to \underset{0}{\pi_n(CX)} \to \pi_n(CX, X) \xrightarrow{\cong} \pi_{n-1}(X) \to \underset{0}{\pi_{n-1}(CX)} \to$$

[provided $n - 1 > 0$]

<u>Pf</u>: <u>Exactness at $\pi_n(X, x_0)$</u>: By compression crit, $j_* \circ i_* = 0$. So $\mathrm{im}\, i_* \subseteq \ker j_*$. Suppose $f : (I^n, \partial I^n) \longrightarrow (X, x_0)$ has $j_*(f) = 0$. Then f is homotopic, rel ∂I^n, to a map into A, i.e. f is in $\mathrm{im}\, i_*$.

<u>Exactness at $\pi_n(X, A, x_0)$</u>: Recall $\partial(f : \square \to X)$ is $f|_{I^{n-1}}$. Thus $\partial \circ j_* = 0$. For $\mathrm{Im}\, j_* \supseteq \ker \partial$, Suppose $f : (I^n, I^{n-1}, J^{n-1}) \longrightarrow (X, A, x_0)$ has $f|_{I^{n-1}} = 0$ in $\pi_n(A, x_0)$. Let F be the homotopy from $f|_{I^{n-1}}$ to const_{x_0}



So $j_*([g]) = [f]$ as desired. $\blacksquare$

<u>Whitehead's Thm</u>: $f: X \to Y$ a map between connected CW complexes. If f_* induces an $\cong$ on each π_n, then f is a homotopy equivalence. Moreover, if f is the inclusion of a subcomplex, then Y deformation retracts to X.

<u>Cor.</u> If Y is a CW complex with $\pi_n Y = 0$ for all n, then X is contractible.

<u>Pf</u>: Take $X = \{pt\}$ in Y.

[To prove Whitehead's Thm, will need]

<u>Compression Lemma</u>: $(X, \overset{\text{sub cplx}}{A})$ a CW pair and (Y, B) any pair with $B \neq \emptyset$. For each n such that $X \setminus A$ has a cell of dim n, assume that $\pi_n(Y, B, y_0) = 0$ for <u>all</u> $y_0 \in B$. Then every $f: (X, A) \to (Y, B)$ is homotopic, rel A, to a map $X \to B$.

<u>Note</u>: When $n = 0$, the cond. should be interpreted as saying that (Y, B) is path connected.

Pf: Assume by induction that $f(X^{k-1}) \subseteq B$.

[Discuss base case] Suppose $\Phi : D^k \longrightarrow X$

is the characteristic map of a cell e^k in $X \setminus A$.

Then $f \circ \Phi : (D^k, \partial D^k) \longrightarrow (Y, B)$. Since

$\pi_k(Y, B, \text{any pt}) = 0$, by the compression critrion

$f \circ \Phi$ is homotopic to a map into B. Doing

this for all k-cells at once gives a homotopy

of $f|_{X^k \cup A}$ to a map to B. By the

homotopy extension property, this extends to a

homotopy to a map of all of X to Y which

is constant on $X^k \cup A$ and so in particular sends
it into B.

If X has cells of arbitrarily high dimension,

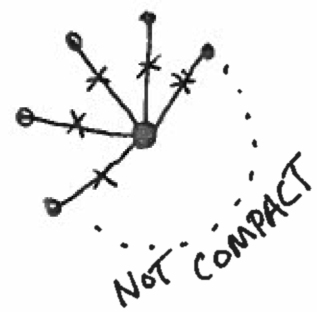
perform the k^{th} homotopy in time $[1-2^{-k}, 1-2^{-(k+1)}]$.

This is a continuous map of $X \times I \longrightarrow Y$ since

it is const on X^k for $t \in [1-2^{-(k+1)}, 1]$. $\blacksquare$

Topology of CW complexes: [Only if comes up.]

1. Start with discrete set X^0.
2. Form X^n from X^{n-1} by attaching n-cells D^n_α via maps $\varphi_\alpha : S^{n-1} \to X^{n-1}$. So X^n is
$$X^{n-1} \coprod_\alpha D^n_\alpha \Big/ {\scriptstyle x \sim \varphi_\alpha(x) \text{ for } x \in \partial D^n_\alpha}$$
with the quotient topology.
3. $X = \bigcup_n X^n$ has the weak topology. A set U is open in X iff $U \cap X^n$ is open for all n.



Prop: A compact $A \subseteq X$ is contained in a finite subcomplex.

Pf idea: Take $S = \left\{ \begin{array}{l} \text{some pt in} \\ \mathrm{int}(e^n_\alpha) \cap A \end{array} \;\middle|\; \mathrm{int}(e^n_\alpha) \cap A \neq \emptyset \right\}$

This is a closed discrete subset of a cpt set A, hence finite. So A only meets finitely many cells, and you can inductively turn this into a subcomplex.