Lecture 21: Relative homotopy groups

$$\pi_n(X, x_0) = \begin{matrix}\text{homotopy} \\ \text{classes} \\ \text{of maps}\end{matrix} \quad (S^n, s_0) \longrightarrow (X, x_0)$$

Thm. A covering map $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ induces $\cong$ on π_n for $n \geq 2$.

Cor: If the universal cover of X is contractable, then $\pi_n X = 0$ for all $n \geq 2$.

Ex: S^1, , $T^n = (S^1)^n$

Thm X_α path conn. Then $\pi_n\left(\prod_\alpha X_\alpha\right) = \prod_\alpha \pi_n(X_\alpha)$.

[Compare with Kunneth theorem]

Pf: A cont map $f: S^n \to \prod_\alpha X_\alpha$ is the same as $\{f_\alpha : S^n \xrightarrow[\text{cont}]{} X_\alpha\}$ and a homotopy between such is the same as $\{F_\alpha : S^n \times I \to X_\alpha\}$.

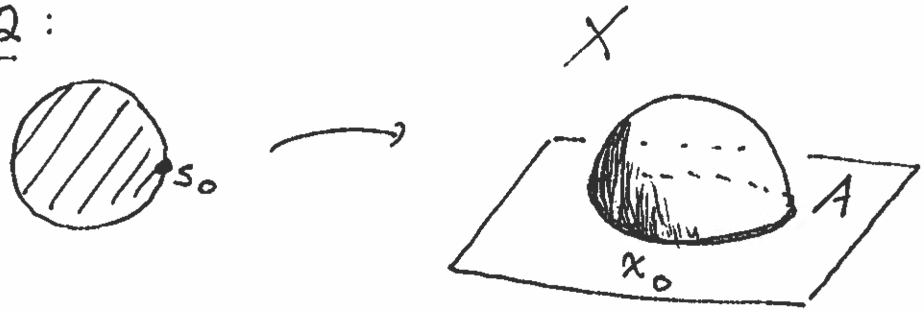
$f_\alpha = p_\alpha \circ f$

Can manipulate each coordinate independently, so done. $\blacksquare$

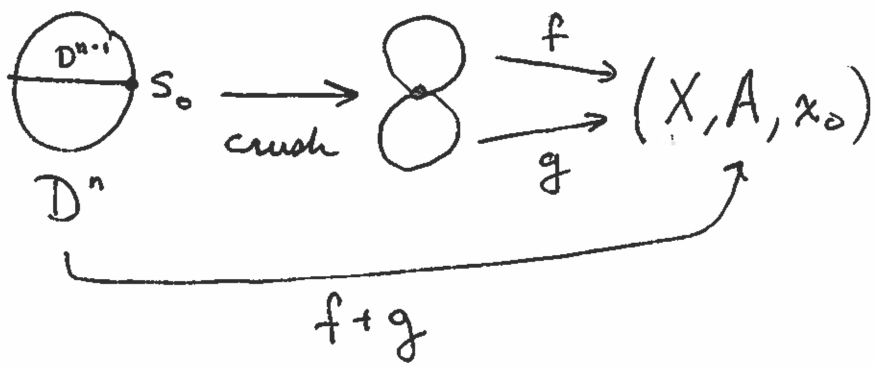
Relative homotopy groups: $X \supseteq A \ni x_0$

$$\pi_n(X, A, x_0) = \text{homotopy classes of maps } (D^n, S^{n-1}, s_0) \longrightarrow (X, A, x_0)$$

$n=2$:



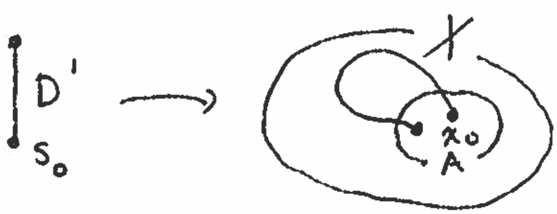
Have gp op: $D^n \xrightarrow[D^{n-1}]{\text{crush}} D^n \vee D^n \longrightarrow X$



Note: Taking $A = \{x_0\}$ gives usual $\pi_n(X, x_0)$

Special cases: **$n=0$:** doesn't make sense

$n=1$:



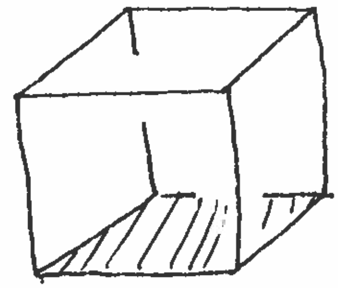
$\pi_1(X, A, x_0)$ = homotopy classes of paths starting at x_0 and ending in A.

No group law.

$n=2$: Group str may not be abelian

Prop: $\pi_n(X, A, x_0)$ is abelian when $n \geq 3$.

Alt description:

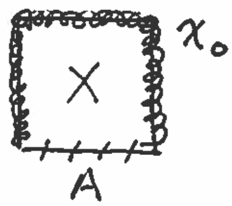


$(I^n, \partial I^n, J) \to (X, A, x_0)$

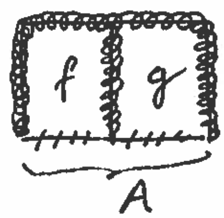
$I^n \supseteq I^{n-1} = \{s_n = 0\}$

$J = \partial I^n \setminus \mathrm{int}(I^{n-1})$

$n = 2$ example.

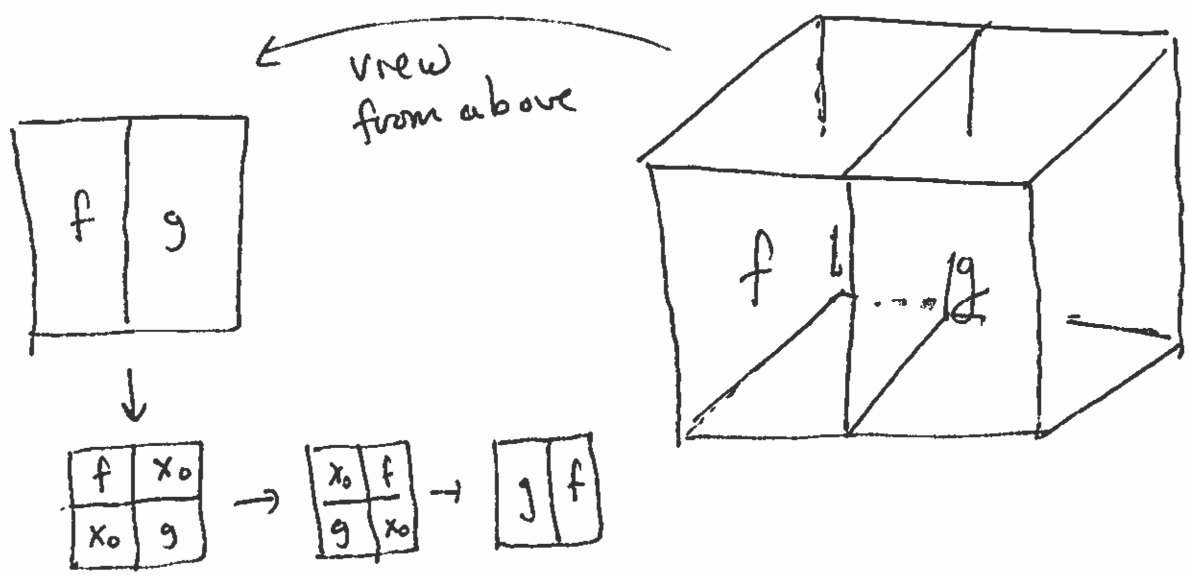


$f + g =$



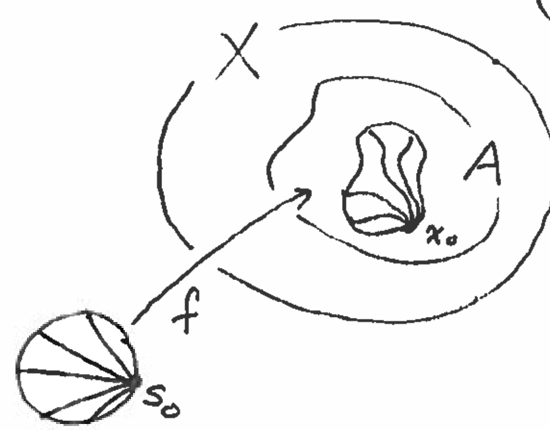
[I^{n-1} is special in that it doesn't have to go to the base pt]

Commutativity trick still works w/ $n \geq 3$.



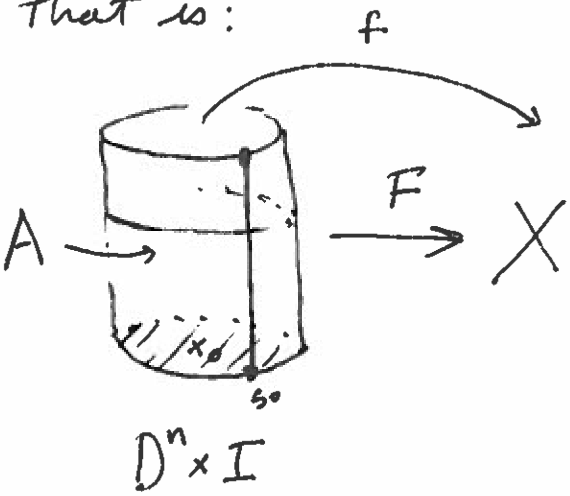
Compression Criterion: $f: (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ is 0 in π_n iff it is homotopic, $\underbrace{\text{rel } S^{n-1}}$, to a map with image in A.

means that $f_t|_{S^{n-1}}$ fixed throughout the homotopy.

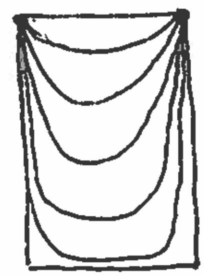


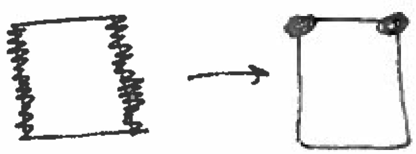
<u>Pf</u>: ($\Leftarrow$) If $f(D^n) \subseteq A$ can homotope to a const map via:

($\Rightarrow$) Suppose $f = 0$ in π_n, that is:



Consider this family of discs in $D^n \times I$ to get a homotopy of f into A with $f|_{S^{n-1}}$ fixed.



[Can think about via: ]

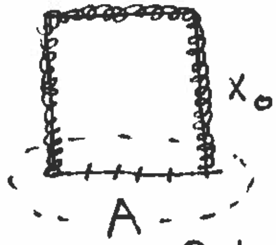
Long <u>exact</u> <u>sequence</u>.

$$\longrightarrow \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \longrightarrow \pi_{n-1}(X, x_0) \longrightarrow \cdots$$

$$\longrightarrow \underbrace{\pi_1(X, A, x_0) \longrightarrow \pi_0(A, x_0) \longrightarrow \pi_0(X, x_0)}$$

not groups, but exactness still makes sense because each has a distinguished elt, namely the const. map to x_0

$\pi_n(X, A, x_0)$



Restrict to here.

$$\partial\big(f : (D^n, S^{n-1}, s_0) \to (X, A, x_0)\big) = f|_{S^{n-1}}$$

$$\partial\big(f : (I^n, \partial I^{n-1}, J) \to (X, A, x_0)\big) = f|_{I^{n-1}}$$

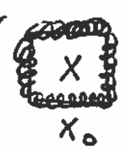
Proof: [Special cases are an exercise.]

Exactness at $\pi_n(X, x_0)$: By compression crit, have $j_* \circ i_* = 0 \Rightarrow \operatorname{Im} i_* \subseteq \operatorname{Ker} j_*$. Suppose $j_*(f) = 0$

$\Rightarrow f$ can be homotoped into A keeping $\underbrace{f|_{\partial I^n}}_{= \text{const}_{x_0}}$ fixed.

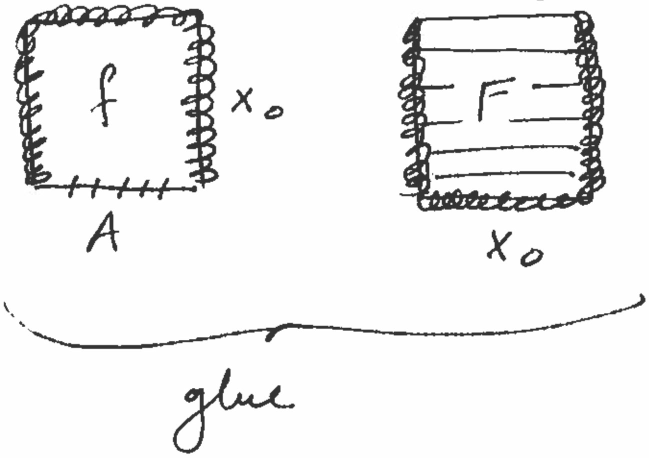
Thus $f \in \operatorname{Im}(i_*)$.

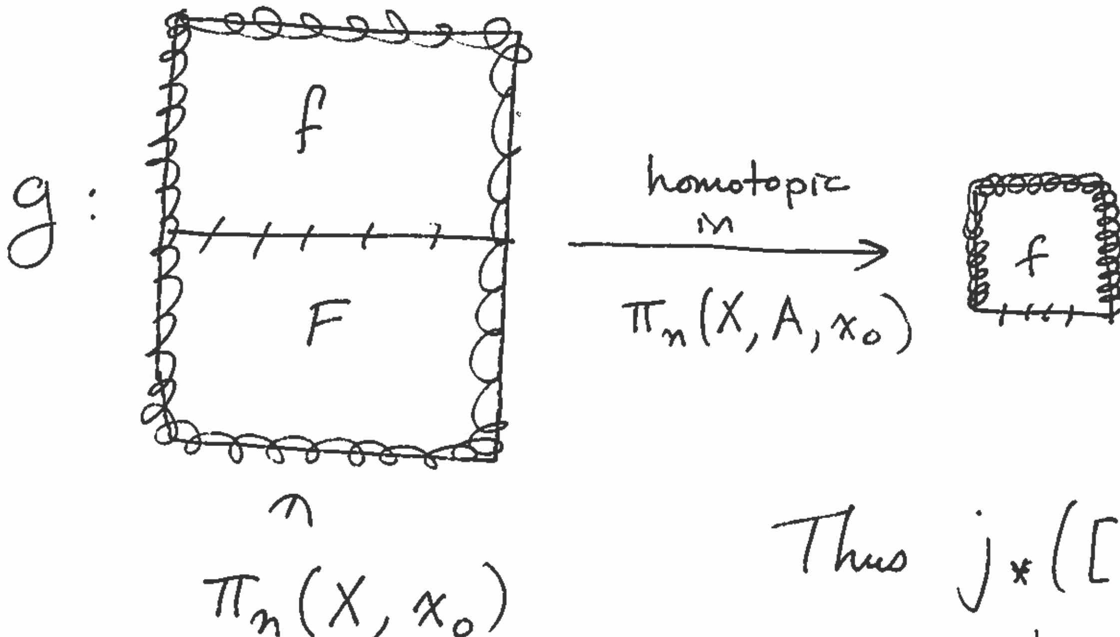
Exactness at $\pi_n(X, A, x_0)$: $\partial \circ j_* = 0$ as

$\partial\Big($  $\Big) =$ restriction of f to  $= \text{const}_{x_0}$. For $\operatorname{Im} j_* \supseteq \operatorname{Ker} \partial$,

suppose $f : (I^n, \partial I^n, J) \longrightarrow (X, A, x_0)$ has $f|_{I^{n-1}} = 0$ in $\pi_n(A, x_0)$. Let F be the homotopy from $f|_{I^{n-1}}$ to const.

$f|_{I^{n-1}} : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$





Thus $j_*([g]) = [f]$ as desired.

[The remaining case is similar, is an exercise.]. ▨