# Lecture 36: More on fiber bundles

Fix a ring $R$, set $K_n = K(R,n)$

**Claim:** $\pi_{m+n}\left(K_m \wedge K_n = K_m \times K_n / K_m \vee K_n\right) = R \otimes_{\mathbb{Z}} R$

**Pf:** Same as $\tilde{H}_{m+n}(K_m \wedge K_n ; \mathbb{Z})$ by Hurewicz. By full Künneth theorem for smash products (Hatcher pg. 276), have

$$0 \to \underbrace{\bigoplus_i \left(\tilde{H}_i(K_m) \otimes \tilde{H}_{m+n-i}(K_n)\right)}_{\text{only non-zero term is } \tilde{H}_m(K_m) \otimes \tilde{H}_n(K_n)} \to \tilde{H}_{n+m}(K_m \wedge K_n)$$

$$\to \bigoplus_i \underbrace{\mathrm{Tor}\left(\tilde{H}_i(K_m), \tilde{H}_{n+m-i-1}(K_n)\right)}_{\text{all } 0.} \to 0$$

[Point: with reduced (co)homology, $X \wedge Y$ is more nat'l than $X \times Y$.]

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**Topological group:** A topological space with a group structure where multiplication $G \times G \to G$, $(g, h) \mapsto g \cdot h$ and inversion $G \to G$, $g \mapsto g^{-1}$ are continuous.

**Ex:** Lie groups: $GL_n \mathbb{R}$, $O(n)$

Discrete groups: Any $G$ with the discrete topology (every set is open)

$$\mathbb{Z}_p = \varprojlim \left( \mathbb{Z}/p^n\mathbb{Z} \leftarrow \mathbb{Z}/p^{n+1}\mathbb{Z} \right)$$

Such a $G$ acts on a space $X$ via a continuous map $G \times X \longrightarrow X$ obeying $g\cdot(h\cdot x) = (gh)\cdot x$ and $1_G \cdot x = x$. Equivalently, an action is a continuous homomorphism $G \longrightarrow \underbrace{\mathrm{Homeo}(X)}_{\text{cpt-open topology}}$.

<u>Ex</u>: $GL_n\mathbb{R}$ acts on $\mathbb{R}^n$ via linear transformations.

$G$ acts on itself by left multiplication

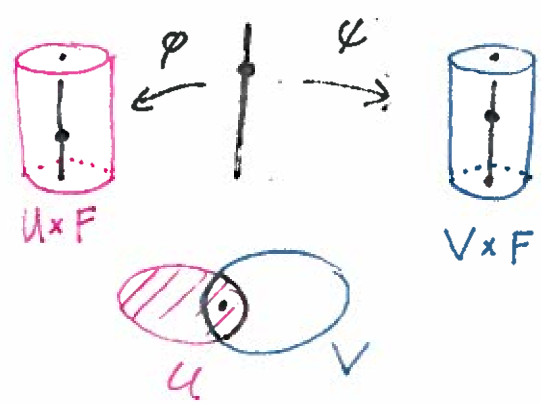
<u>Def</u>: Suppose $G$ is a topological group acting on a space $F$. A <u>fiber bundle $E$ over $B$ with fiber $F$ and structure group $G$</u> is a map $p: E \longrightarrow B$ together with a collection of homeos $\{\varphi : p^{-1}(U) \longrightarrow U \times F\}$ where $U \subseteq B$, called charts, where

1) The sets $U$ cover $B$.

2) Each diagram
$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\varphi"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & U \times F \arrow[dl, "\mathrm{proj}_U"] \\ & U & \end{tikzcd}$$
commutes.

3) If $(U,\varphi)$ and $(V,\psi)$ are both charts, then $\exists$ a continuous map $\theta : U \cap V \longrightarrow G$ so that $\forall x \in U \cap V$ and $f \in F$ we have

$$\theta(x)\cdot f = \varphi(\psi^{-1}((x,f)))$$



Ex: (1) If $G = \mathrm{Homeo}(F)$, then this is just our notion of a fiber bundle from before.

(2) $G = GL_n\mathbb{R}$, $F = \mathbb{R}^n$ gives a vector bundle. In particular, each $p^{-1}(b)$ is a vector space of dim $n$. [While many diff. ids of $p^{-1}(b)$ with $\mathbb{R}^n$ can define vector addition in any of them; alternatively, think of as an old style fiber bundle with extra structure.).

a) $M^n$ smooth, then $TM \to M$ is a vector bundle.

b) If $g$ is a Riemannian metric, then $TM \to M$ has fiber $\mathbb{R}^n$ and structure group $O(n)$.

(3) Principal bundle: $F = G$ acted on by left-translation.

Construction: Given a homomorphism $\pi_1 B \xrightarrow{\rho} G$ (connected)
consider $\tilde{E} = \tilde{B}_{univ} \times G$ with the $\pi_1 B$ action

$$\gamma \cdot (\tilde{b}, g) = (\gamma \cdot \tilde{b}, \rho(\gamma) g)$$

action as covering translation

Set $E = \pi_1 B \backslash \tilde{E}$ which has a map $p : E \to B$ namely $[(\tilde{b}, g)] \longmapsto \pi(\tilde{b})$

where $\pi: \tilde{B}_{univ} \to B$ is the covering map. Note $\tilde{E} \xrightarrow{q} E$ is itself a covering map.

Why this is a fiber bundle: For each evenly covered connected $U \subseteq B$ and component $\tilde{U}$ of $\pi^{-1}(U)$ define a chart $\varphi: p^{-1}(U) \longrightarrow U \times G$ as the inverse of

$$U \times G \xrightarrow[\pi^{-1} \times id]{} \tilde{U} \times G \xrightarrow[q]{} p^{-1}(U)$$

This is a homeomorphism since if $(\gamma \cdot \tilde{U}) \cap \tilde{U}$ then $\gamma = 1_{\pi_1 B}$.

Allows us to build <u>many</u> bundles.

$B =$  $\pi_1 B \twoheadrightarrow F_2 = \langle x, y \rangle$

now pick any $X, Y \in SU(2)$ to get a 5-manifold $E_{X,Y} \to B$.

> <u>Ex</u>: If $G$ is discrete, then a principal $G$ bundle is a regular covering space corresponding to a homomorphism $\pi_1 B \to G$.

<u>Technical notes</u>: In definition, should add

4) If $(U, \varphi)$ is a chart and $V \underset{open}{\subseteq} U$ then $(V, \varphi|_V)$ is also a chart.

5) The collection $(U, \varphi)$ is maximal with respect to (1-4)

6) $G$ acts effectively on $F$, i.e. $G \longrightarrow Homeo(F)$

Pull backs: Suppose $f: A \to B$ and $p: E \to B$ a fiber bundle. Define $f^*(E) = \{(a,e) \in A \times E \mid f(a) = p(e)\}$

Have

$$\begin{tikzcd} f^*(E) \arrow[r, "\pi_E"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Exercise: Prove that $f^*(E) \xrightarrow[\pi_A]{} A$ is a fiber bundle with the same fiber and structure group as $E \to B$.

Universal Bundles: If $G$ is a topological group there exists a principal $G$-bundle $EG \to BG$ so that for any CW complex $X$

$$[X, BG] \simeq \begin{array}{c}\text{Isomorphism}\\ \text{classes of}\\ \text{principal } G\text{-bundles}\\ \text{over } B\end{array}$$

$$(f: X \to BG) \longmapsto f^*(EG).$$

If $G$ has the discrete topology, then

$BG = K(G,1)$ and $EG \to BG$ is as in the 1st proof of the existence of $K(G,1)$'s.