Lecture 35:

Ω-spectrum: A sequence of CW complexes $\{K_n\}$ together with homotopy equivalences $K_n \to \Omega K_{n+1}$.

Thm: If K_n is an Ω-spectrum, then $h^n(X) = \langle X, K_n \rangle$ defines a reduced cohomology theory of based CW complexes.

Brown Representability: Any reduced cohomology theory of CW complexes comes from some Ω-spectrum.

Thm. G abelian group. There is a class $\alpha_n \in H^n(K(G,n); G)$ so that

$$T : \langle X, K(G,n) \rangle \longrightarrow H^n(X; K)$$
$$f \longmapsto f^*(\alpha_n)$$

is an isomorphism for all CW complexes X.

Pf. Let K_n be the Eilenberg-MacLane Ω-spectrum. By theorem, this gives a cohomology theory h^n.

Now $h^*(S^0) = \begin{cases} 0 & * \neq 0 \\ \langle S^0, K_0 \rangle \cong G & * = 0 \end{cases}$

s_0

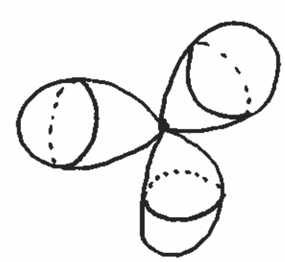
discrete set of pts of size G

where the group structure on $\langle S^0, K_0 \rangle$ comes from the homotopy equivalence $K_0 \simeq \Omega K_1$. By uniqueness of cohomology for CW complexes with this value for $h^*(S^0)$, have $h^*(X) \cong H^*(X; G)$ via a <u>natural</u> isomorphism T.

Define α_n as the image of $id_{K(G,n)}$ under T. Then for $f \in \langle X, K(G,n) \rangle$ we have

$$T([f]) = T([id_{K(G,n)} \circ f]) = T(f^*([id_{K(G,n)}]))$$
$$= f^* T([id_{K(G,n)}]) = f^*(\alpha_n).$$

<u>Geometric construction:</u> K_n a $K(G,n)$ with $K_n^{(n-1)} = pt$. Then α is the cellular cochain assigning to an n-cell e_α the corresp. elt of $\pi_n K_n = G$.



[Query: Why is $\delta\alpha = 0$?]

<u>Cup product:</u> $X \xrightarrow[f]{} K_n \qquad Y \xrightarrow[g]{} K_m$ where $G = R$ a ring.

$K_n \wedge K_m =$ smash product $= K_n \times K_m \Big/ \underbrace{K_n \vee K_m}_{\text{becomes base point}}$ ← wedge at basepoints

which is $n+m-1$ connected

Ex:
$S^1 \wedge S^1 = S^2$
$S^n \wedge S^m = S^{n+m}$

So $\pi_{n+m}(K_n \wedge K_m) \cong H_{n+m}(K_n \wedge K_m; \mathbb{Z}) =$

$$H_n(K_n) \otimes_{\mathbb{Z}} H_m(K_m) = R \otimes_{\mathbb{Z}} R \xrightarrow{\text{mult}} R$$

Define $K_n \wedge K_m \xrightarrow{\mu} K_{n+m}$

so that μ_* on π_{n+m} is the mult map in R.

Then the cross product $[f] \times [g] \in H^{n+m}(X \times Y)$

is the composition

$$X \times Y \xrightarrow[f \times g]{} K_n \times K_m \longrightarrow K_n \wedge K_m \xrightarrow{\mu} K_{n+m}.$$

If $g : X \longrightarrow K_m$, then $[f] \cup [g]$ is

the composition $X \xrightarrow[\Delta]{} X \times X \xrightarrow[f \wedge g]{} K_n \wedge K_m \xrightarrow[\mu]{} K_{n+m}$.

Can check the basic props of cup product this way.

Sign comes down to $S^m \wedge S^n \to S^n \wedge S^m$ has degree $(-1)^{nm}$.

That this is really the cup product follows from confirming this for the α_n in $H^n(K(G,n);G)$.

<u>Cohomology of Fiber Bundles</u> (Hatcher 4.D)

$$F \longrightarrow E \xrightarrow{p} B$$

[Recall complexities of Künneth Thm...]

[Ask about vector bundles, smooth mflds.]

<u>Leray-Hirsch</u>: Let $F \xrightarrow[i]{} E \xrightarrow[p]{} B$ be a fiber bundle and R a ring. Suppose

ⓐ $H^n(F;R)$ is a finitely generated free R-module for all n.

ⓑ $\exists\, c_j \in H^{k_j}(E;R)$ where $i^*(c_j)$ form a basis for $H^*(F;R)$

Then
$$\Phi : H^*(B;R) \otimes_R H^*(F;R) \longrightarrow H^*(E;R)$$
$$b \otimes i^*(c_j) \longmapsto p^*(b) \cup c_j$$

is an isomorphism of R-modules. [<u>But</u>: May not be a ring isomorphism]

Ex where L-H applies: $E = B \times F$.

Ex where it doesn't: Hopf bundle $S^1 \to S^3 \to S^2$.

[Will do proof next time. For now, here's a concept we'll need.]

<u>Pull back bundle</u>: Suppose have $f : A \longrightarrow B$.

Define $f^*(E) = \{ (a,e) \in A \times E \mid f(a) = p(e) \}$

Have

$$\begin{tikzcd} f^*(E) \arrow[r, "\pi_e"] \arrow[d, "\pi_a"] & E \arrow[d, "p"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Note that $f^*(E) \xrightarrow[\pi_a]{} A$ is a fiber bundle with fiber F. Local triviality follows from observing that if p is trivial over U then π_a is trivial over $f^{-1}(U)$.