Lecture 34: Ω-spectra give cohomology theories.

Ω-spectrum: A sequence of CW complexes $\{K_n\}$ together with homotopy equivalences $K_n \to \Omega K_{n+1}$

Eilenberg-Maclane Spectrum: Fix abelian G, take $K_n = K(G, n)$ for $n \geq 0$ where $K(G,0)$ is just a discrete set of G points.

Thm If K_n is an Ω-spectrum, then $h^n(X) = \langle X, K_n \rangle$ defines a reduced cohomology theory of based CW complexes.

Note: If h^n is a reduced based theory, then $\bar{h}^n(X) = h^n(X_+ = X \text{ with disjoint base pt})$ is an unreduced based theory.

Proof: ① h^n is a functor $CW_{\text{based}} \longrightarrow \mathcal{A}b$ and f^* depends only on the homotopy class of $f: X \to Y$.

Define $f^*: \langle Y, K_n \rangle \longrightarrow \langle X, K_n \rangle$ by

$(g : Y \to K_n) \longmapsto (g \circ f : X \to K_n)$ which clearly depends only on the (based!) homotopy class of f.

Also f^* is a homomorphism in the gp str coming from $K_n = \Omega K_{n+1} = \Omega^2 K_{n+2}$.

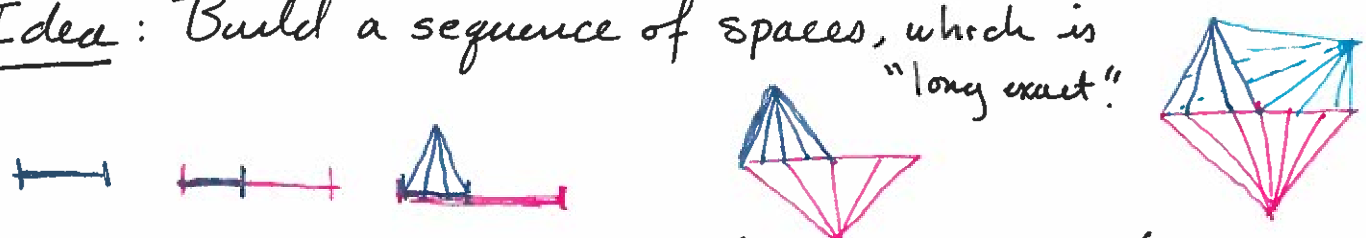
3 $h^n(\bigvee_\alpha X_\alpha) = \prod_\alpha h^n(X_\alpha)$

$$\left\{ \begin{array}{l} \text{base pt pres} \\ \text{maps } \bigvee_\alpha X_\alpha \to K_n \end{array} \right\}_{\big/ \text{homotopy}} \cong \left\{ \begin{array}{l} \text{A collection of based} \\ \text{maps } X_\alpha \to K_n \end{array} \right\}$$

2 Long exact sequence of (X, A) subcomplex.

$$\leftarrow h^n(A) \xleftarrow{i^*} h^n(X) \xleftarrow{q_*} h^n(X/A) \underset{\delta}{\leftarrow} h^{n-1}(A) \leftarrow$$

<u>Idea</u>: Build a sequence of spaces, which is "long exact"



$\star$ $A \hookrightarrow X \hookrightarrow X \cup CA \hookrightarrow (X \cup CA) \cup CX \hookrightarrow ((X \cup CA) \cup CX) \cup C(X \cup CA)$

$$\begin{array}{ccccccccc} & & & & \wr\!\wr_{h.e} & & \wr\!\wr_{h.e.} & & \\ A & \hookrightarrow & X & \longrightarrow & X/A & \xrightarrow[\text{[fuzzy]}]{} & \Sigma A & \hookrightarrow & \Sigma X \end{array}$$

Note: Really using the reduced cone here.

On the top row, the pattern is to add the cone over the stage two steps before. Bottom row continues:

$$\longrightarrow \Sigma X/A \longrightarrow \Sigma^2 A \longrightarrow \Sigma^2 X \longrightarrow \Sigma^2(X/A) \longrightarrow$$

sequence suspended up.

Take homotopy classes of maps to a space K

$$\overbrace{\langle A, K\rangle \longleftarrow \langle X, K\rangle \longleftarrow \langle X/A, K\rangle}^{\text{not groups but have dist. elements}} \longleftarrow$$

$$\longleftarrow \langle \Sigma A, K\rangle \longleftarrow \langle \Sigma X, K\rangle \longleftarrow \langle \Sigma(X/A), K\rangle$$

groups from here on out.
abelian groups starting at $\langle \Sigma^2 A, K\rangle$.

<u>Claim</u>: This sequence is exact. [Prove in a minute]

Take $K = K_n$. Then we have

$$h^n(A) \leftarrow h^n(X) \leftarrow h^n(X/A) \leftarrow h^{n-1}(A) \leftarrow$$

$$\begin{array}{ccccccc}
\langle A, K_n\rangle & \leftarrow & \langle X, K_n\rangle & \leftarrow & \langle X/A, K_n\rangle & \leftarrow & \langle \Sigma A, K \\
\cong & & & & & & \cong \\
\langle A, \Omega K_{n+1}\rangle & \leftarrow & \langle X, \Omega K_{n+1}\rangle & \leftarrow & \langle X/A, \Omega K_{n+1}\rangle & & \langle A, \Omega K \\
\cong & & & & & & \cong \\
 & & & & & & K_{n-1}
\end{array}$$

$$\begin{array}{ccccccccc}
\langle X, K_{n+1}\rangle & \leftarrow & \langle X/A, K_{n+1}\rangle & \leftarrow & \langle \Sigma A, K_{n+1}\rangle & \leftarrow & \langle \Sigma X, K_{n+1}\rangle & \leftarrow \\
 & & \cong & & & & & \\
h^{n+1}(X) & \leftarrow & h^{n+1}(X,A) & & & & &
\end{array}$$

giving the needed long exact sequence, which respects $(X,A) \xrightarrow[g]{} (Y,B)$ since the construction ($\star$) does.

Proof of claim: Note that in ($\star$) every three terms $\underbrace{}_{}$ starting anywhere have the form $B \hookrightarrow Y \hookrightarrow Y \cup CB$. So E.T.S. exactness at the middle term in

$$\langle A, K\rangle \leftarrow \langle X, K\rangle \leftarrow \langle X \cup CA, K\rangle$$

If $f|_A \simeq \text{const}_{k_0} \longleftarrow f : X \to K$ then clearly f extends over CA to give a map in