Framed Bordism.

Last time: M, N closed smooth n-manifolds are bordant if $\exists$ smooth W^{n+1} with ∂W diffeom to $M \amalg N$.

$$\Omega_n = \left\{ \begin{array}{l} \text{closed smooth } n\text{-mflds,} \\ \text{up to cobordism.} \end{array} \right\} \quad \left[\begin{array}{l} \text{Addition: Disjoint union} \\ \text{Mult: Product} \end{array} \right]$$

Thom: $\Omega_* = \bigcup \Omega_n$ is a polynomial algebra over $\mathbb{F}_2$

on generators u_i for $i > 1$ and $i \neq 2^r - 1$.

$\uparrow$ degree i

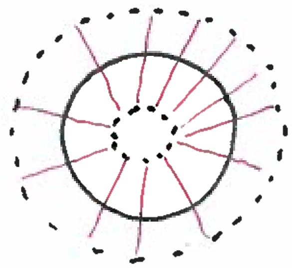
Correction:

$$\Omega_4 = \{ \underset{0}{[\phi]}, \underset{u_4}{[\mathbb{RP}^4]}, \underset{u_2^2}{[\mathbb{RP}^2 \times \mathbb{RP}^2]}, \underset{u_4 + u_2^2}{[\mathbb{RP}^4 \amalg \mathbb{RP}^2 \times \mathbb{RP}^2]}$$

Convention: An embedding $\phi : V \to M$ of smooth manifolds is a topological embedding where $d\phi : T_p V \to T_p M$ is 1-1 for all $p \in V$. The image $\phi(V)$ is a submanifold.

Def: A framing of a submanifold $V^{k-n} \subseteq M^k$ is an embedding $\Phi : V \times \mathbb{R}^n \to M$ with $\Phi(v, 0) = v$.

Ex:



$S^1 \times \mathbb{R}$

$S^1 \subseteq \mathbb{R}^2$

Ex: V and M orientable
$n = \text{codim} = 1$.

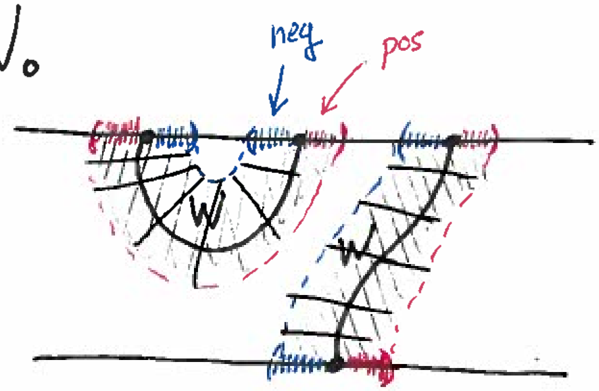
Non Ex: $\mathbb{C}P^1 \subseteq \mathbb{C}P^2$
cannot be framed since:

ⓐ Any perturbation of $\mathbb{C}P^1$ intersects the original once algebraically, corresponding to cup product structure on $H^*(\mathbb{C}P^2; \mathbb{Z})$.

ⓑ Any framed submanifold can be made disjoint from itself: V vs $\Phi(V \times e_1)$

Two framed submanifolds V_0^{k-n} and V_1^{k-n} are framed cobordant if $\exists$ a framed submfld $W^{k-n+1} \subseteq M \times I$ so that $W \cap M \times \{0\} = V_0$ and $W \cap M \times \{1\} = V_1$ as framed manifolds.

V_0



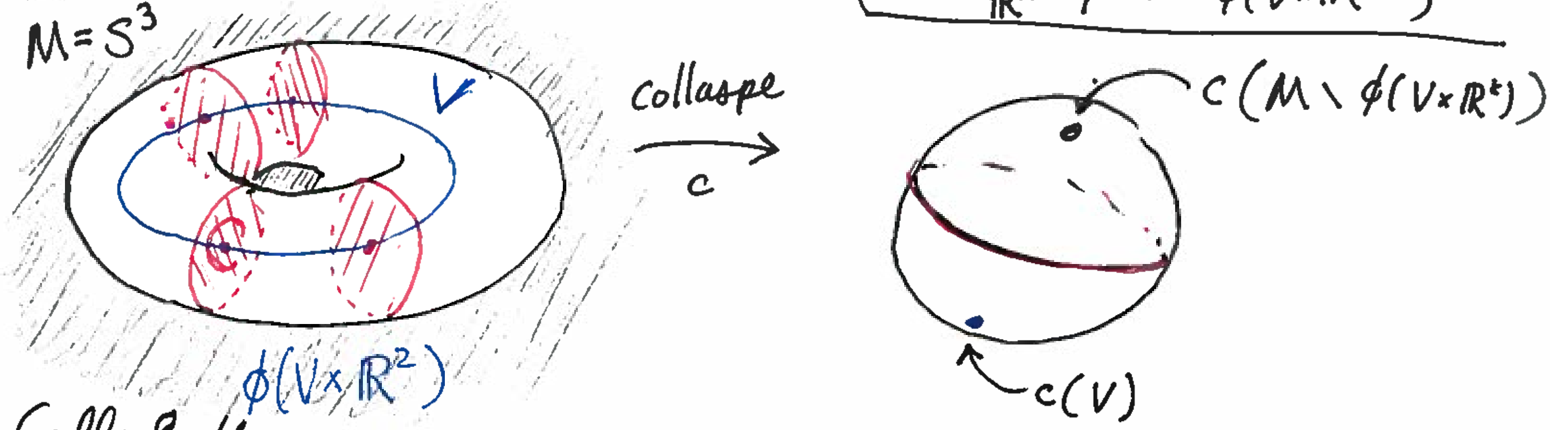
$M \times \{0\}$

V_1

Let $\Omega^{fr}_{k-n,M} = \left\{ \begin{array}{l} \text{framed compact submflds } V^{n-k} \subseteq M \\ \text{without boundary} \end{array} \right\}$ / framed bordism.

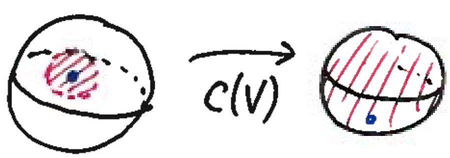
Thm: Suppose M^k is closed. Then there is a bijection $c: \Omega^{fr}_{k-n,M} \longrightarrow [M, S^n]$

given by $(\phi : V \times \mathbb{R}^k \hookrightarrow M) \longmapsto \left(\begin{array}{l} \text{collaspe map to } S^n = \mathbb{R}^n \cup \{\infty\} \\ M \setminus \phi(V\times\mathbb{R}^k) \longmapsto \infty \\ \pi_{\mathbb{R}^k} \circ \phi^{-1} \text{ on } \phi(V \times \mathbb{R}^k) \end{array} \right)$



Called the Pontrjagin-Thom construction. $[M, S^n]$ is a "cohomotopy group".

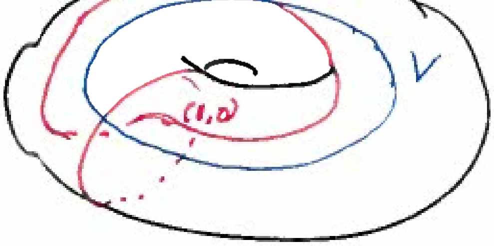
Ex: ① pt $\subseteq S^k$ gives $c(V) \in [S^k, S^k]$ which



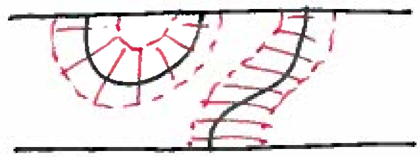
is a gen of $\pi_k(S^k)$.

② $S^1 \subseteq S^2$ is framed bordant to ϕ; gives trivial map in $[S^2, S^1]$.

③ $S^1 \subseteq S^3$ where the framing is such that $\phi(S^1 \times \{(1,0)\})$ links V once. then $c(V) \in [S^3, S^2]$ is the gen of $\pi_3 S^2$.



Why c takes bordant submanifolds V_0, V_1 to homotopic maps:



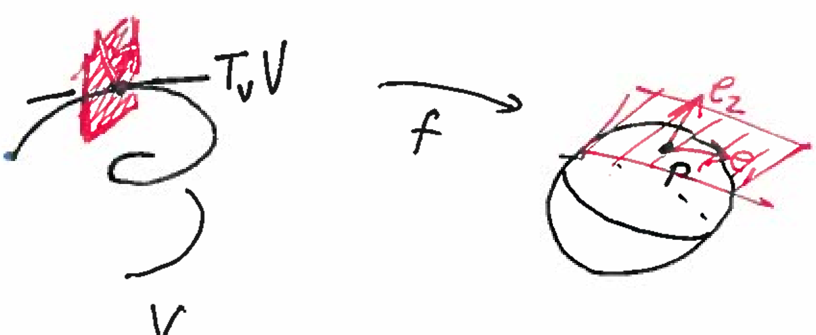
The framed $W \in M \times I$ has its own P-T map: $M \times I \to S^n$ which restricts to the PT map on $M \times \{0\}$ and $M \times \{1\}$.

<u>Defining the inverse:</u> $d : [M, S^n] \to \Omega^{fr}_{n-k, M}$

Let $f : M \to S^n$; we can assume f is smooth.

Pick $p \in S^n$ a regular value (exist by Sard's thm) so that $V = f^{-1}(p)$ is an embedded submanifold and $df_v : T_v M \to T_p S^n$ is onto for all $v \in V$.

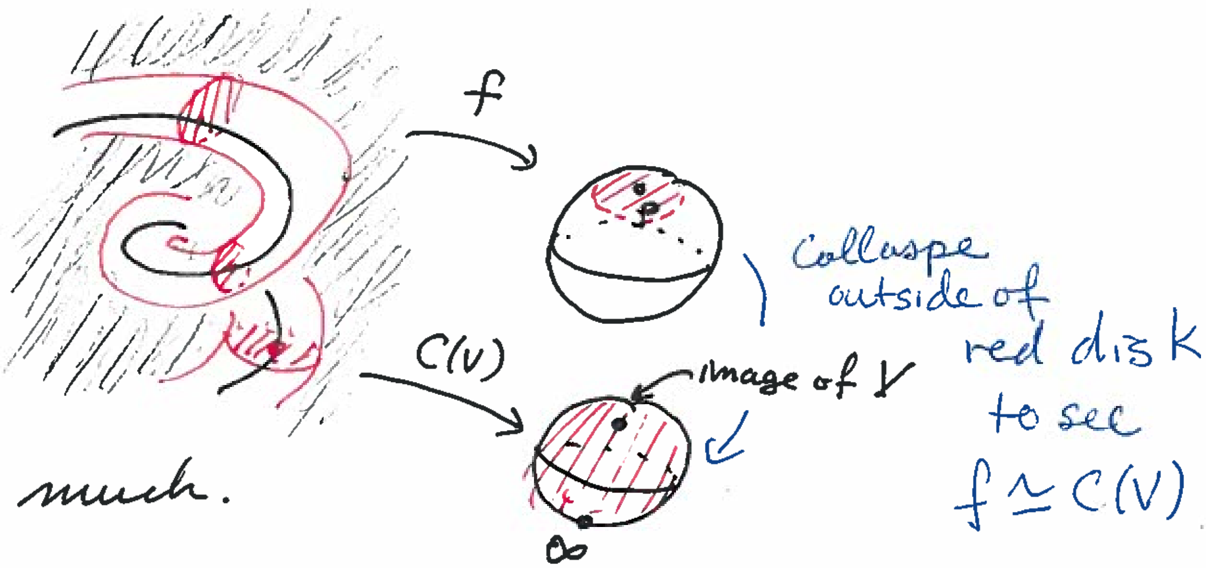
The submfld V gets a framing from $T_p S^n$ by first trivializing the normal bundle to V.



Not so hard to see that homotopic maps give bordant V.

That $d \circ c$ is the identity is clear; just choose $p \in S^n$ used in d to be the image of V.

The other direction $c \circ d$ is a little harder



but not much.

For details, see Chapter 8 of Davis + Kirk. "Lecture notes in algebraic topology."