Homology with local coefficients: (Hatcher §3.H)

G topological group acting on a space F.

$p: E \to B$ a fiber bundle with fiber F and structure group G.

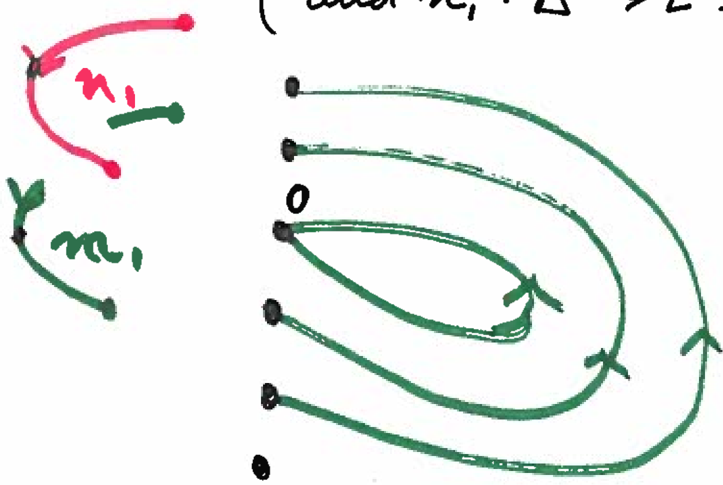
Principal bundle $F = G$ and G acts by left translation.

Bundle of groups: $p: E \to B$ with fiber a discrete abelian group G and structure group $\mathrm{Aut}(G)$ again with the discrete topology. So $p: E \to B$ is a covering space where each $p^{-1}(b)$ has a group structure where locally $p^{-1}(U) \cong U \times G$ via a homeomorphism that is a group isomorphism on each $p^{-1}(b)$.

Ex: M an n-mfld. For R a ring, have

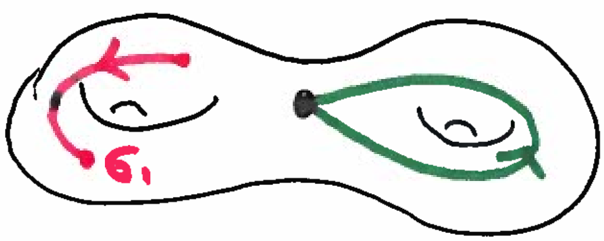
$$\begin{array}{ccc} M_R = \{ & \alpha_x \in H^n(M, M \setminus \{x\}; R) & \mid x \in M\} \\ \downarrow & \big\downarrow & \\ M & x & \end{array}$$

$$C_n(B;E) = \left\{ \begin{array}{l} \text{Finite sums } \sum n_i \sigma_i \text{ where} \\ \sigma_i : \Delta^n \to B \text{ is a singular simplex} \\ \text{and } n_i : \Delta^n \to E \text{ is a lift of } \sigma_i \end{array} \right\}$$



If m_i and n_i are lifts of the same σ_i then define $(m_i + n_i) : \Delta^n \to E$ by

$$(m_i + n_i)(s) = m_i(s) + n_i(s).$$



$C_n(B;E)$ is a group where $m_i \sigma_i + n_i \sigma_i = (m_i + n_i)\sigma_i$. If $E = B \times G$ with $p = \pi_B$ then $C_n(B;E)$ is just $C_n(B;G)$.

Define $\partial : C_n(B;E) \to C_{n-1}(B;E)$ ~~[illegible]~~ by

$$\partial(n_i \sigma_i) = \sum (-1)^j n_i \big|_{[v_0,\ldots,\hat{v}_j,\ldots,v_n]} \sigma_i \big|_{[v_0,\ldots,\hat{v}_j,\ldots,v_n]}$$

As usual, $\partial^2 = 0$ and we can define $H_*(B;E)$, the homology with local coeffs in E. Has all the usual properties...
(twisted)

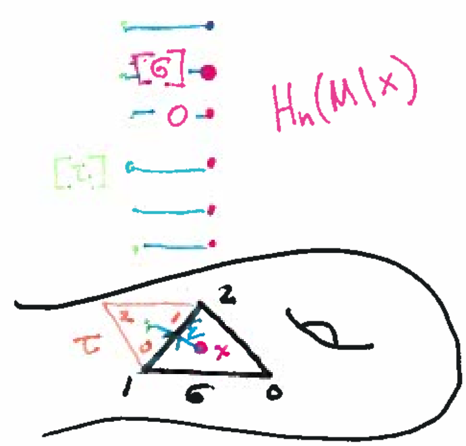
Ex: M^n compact w/o boundary. Then if M is connected then

$$H_n(M; M_{\mathbb{Z}}) = \mathbb{Z}$$

even when M is nonorientable.

[When M is orientable, $M_{\mathbb{Z}} = M \times \mathbb{Z}$ and so the above is just $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$. Regardless of whether M is orientable...]

Suppose $\mathcal{T}$ is a triangulation of M. If $\sigma : \Delta^n \to M$ is one of the n-simplices in M, define a lift $\eta_\sigma : \Delta^n \to M_{\mathbb{Z}}$ by the requirement that $\eta_\sigma(\text{barycenter}) = [\sigma] \in H_n(M \mid \sigma(\text{barycenter}))$



Define $[M] \in H_n(M; M_{\mathbb{Z}})$ by

$$[M] = \sum_{\sigma \text{ in } \mathcal{T}^{(n)} \setminus \mathcal{T}^{(n-1)}} n_\sigma \sigma$$

This actually has $\partial = 0$ by this picture:

Coefficient on ε for $\partial\sigma$ is lift of ~~ε~~ ε ~~at~~ ending at $[\sigma]$

Coefficient on ε for $\partial\tau$ is lift of ε starting at $[\tau]$

Adding these gives 0!

To do Poincaré duality, want a bundle of rings $E \to B$ with fiber R.
with a fundamental class $[M] \in H_n(M; E)$ which restricts to a generator of $H_n(M|x; E) \underset{\text{excision}}{\cong} R$ for all $x \in M$.

Now $\mathbb{Z}$ has no ring automorphisms, so consider $R = \mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\}$. Consider $E \to M$ with fiber R where the monodromy around a non-orientable loop ~~[illegible]~~ results in complex conjugation on R.

Basically $E = (M \times \mathbb{Z}) \oplus (M_{\mathbb{Z}})$ and so
$H_*(M; E) = H_*(M; \mathbb{Z}) \oplus H_*(M; M_{\mathbb{Z}})$ when M is non-orientable. The generator of $H_n(M; M_{\mathbb{Z}})$ is a fundamental class for $H_n(M; E) = 0 \oplus \mathbb{Z} = \mathbb{Z}$ which restricts to a unit ("i") in each $H_n(M|x; E)$.

Capping with this interchanges the real and imaginary parts of $H^*(M; E)$ and so we get.

<u>Thm</u>: M^n compact conn. w/o boundary. If M is non-orientable then $H^k(M;\mathbb{Z}) \cong H_{n-k}(M;M_{\mathbb{Z}})$

$$H^k(M;M_{\mathbb{Z}}) \cong H_{n-k}(M,\mathbb{Z})$$

CW complex.

A bundle of groups over a connected B is specified by a homomorphism $\pi_1 B \xrightarrow{\alpha} \mathrm{Aut}(G)$; equivalently G has the structure of a $\pi = \pi_1 B$-module. Let $\tilde{B}$ = univ cover of B. Then $C_*(\tilde{B})$ is a free $\mathbb{Z}[\pi]$ module; specifically, $C_n(\tilde{B}) = \bigoplus_{\substack{n\text{-cells}\\ \text{of } B.}} \mathbb{Z}[\pi]$. Indeed, the boundary maps for $C_*(\tilde{B})$ are $\mathbb{Z}[\pi]$ module maps. Given a $\mathbb{Z}[\pi]$ module G define

$$C_*(B;G_\alpha) = C_*(\tilde{B}) \otimes_{\mathbb{Z}[\pi]} G$$

where $C_*(\tilde{B})$ has the right $\mathbb{Z}[\pi]$ module structure induced by its left $\mathbb{Z}[\pi]$ module structure: $c\cdot g = g^{-1}c$.

The homology of this is the same as $H_*(B;E_\alpha)$.