Lecture 3: Universal Coefficient Thm

$$C = \{ \to C_{n+1} \to C_n \xrightarrow{\partial} C_{n-1} \to \}$$ chain complex of free abelian gps.

H_n = Homology of C

$H^n(C;G)$ = Cohomology of $\mathrm{Hom}(C_n, G)$ $\quad \leftarrow C^{n+1} \xleftarrow{\delta} C^n \leftarrow C^{n-1}$

Universal Coeff Thm: The following is exact:

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}, G) \longrightarrow H^n(C;G) \xrightarrow[h]{} \mathrm{Hom}(H_n, G) \to 0$$

and moreover splits.

Start with h: $[\varphi] \in H^n(C;G)$ $\qquad \varphi \in C^n = \mathrm{Hom}(C_n, G)$

$[\alpha] \in H_n$ $\qquad \alpha \in C_n$

$$\boxed{\delta\varphi = 0,\ \partial\alpha = 0}$$

Set $h([\varphi])([\alpha]) = \varphi(\alpha) \in G$

Well defined: ⓐ Replacing α with $\alpha + \partial\beta$

$$\varphi(\alpha + \partial\beta) = \varphi(\alpha) + \varphi(\partial\beta) = \varphi(\alpha) + \delta\varphi(\beta) = \varphi(\alpha). \checkmark$$

(b) Replacing φ with $\varphi + \delta\psi$ is OK since

$$\delta\psi(\alpha) = \psi(\partial\alpha) = \psi(0) = 0 \ \checkmark$$

$\blacksquare$

<u>Lemma</u>: h is onto.

<u>Pf</u>: Let $\varphi_0 \in \mathrm{Hom}(H_n, G)$. Consider

FREE ABELIAN

$\ker\partial_n = Z_n$, $\quad \mathrm{im}\,\partial_n = \text{boundaries} = B_{n-1}$

$$\begin{tikzcd} 0 \arrow[r] & Z_n \arrow[r, "i"] & C_n \arrow[r, shift left, "\partial_n"] \arrow[d, phantom, "\cong"] & B_{n-1} \arrow[r] \arrow[l, dashed, shift left] & 0 \\ & & Z_n \oplus B_{n-1} & & \end{tikzcd}$$

Get $\rho : C_n \to Z_n$ where $\rho \circ i = \mathrm{id}|_{Z_n}$. Define

$\varphi_1 : C_n \to G$ by
$$\begin{tikzcd} C_n \arrow[r, "\rho"'] \arrow[rrr, bend right, "\varphi_1"'] & Z_n \arrow[r] & H_n \arrow[r] & G \end{tikzcd}$$
$\in C^n(C; G)$

Now $\delta\varphi_1(\alpha) = \varphi_1(\partial\alpha) = \varphi_0([\partial\alpha]) = 0$

$\alpha \in C_{n+1}$

So $\varphi_1 \in C^n(C; G)$ is a cocycle where $h(\varphi_1) = \varphi_0$.

So h is onto. $\blacksquare$

Ext: The derived functor of Hom$(-, G)$

Def: A <u>free resolution</u> of an abelian gp H is an exact seq

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \to H \to 0$$

where each F_i is free abelian.

Ex: $H = \mathbb{Z} \oplus \mathbb{Z}/2$

(a) $$0 \to 0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}/2 \to 0$$
$$1 \mapsto (0,2)$$

(b) $$\cdots \to \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathbb{Z} \oplus \mathbb{Z}/2 \to 0$$

Def: Ext(H, G) is the first cohomology gp of

$$\leftarrow F_2^* \leftarrow F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0$$

where $A^* = \mathrm{Hom}(A, G)$ and F_i are <u>any</u> free resolution of H.

Easy rules: $\operatorname{Ext}(H \oplus H', G) = \operatorname{Ext}(H, G) \oplus \operatorname{Ext}(H', G)$

If H is free $\operatorname{Ext}(H, G) = 0$ $\quad [0 \to 0 \to H \xrightarrow{\cong} H \to 0]$

$\operatorname{Ext}(\mathbb{Z}/n, G) = G/nG$. $\quad [0 \to \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \xrightarrow{} \mathbb{Z}/n \to 0]$

$\operatorname{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2 \qquad \operatorname{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/3) = 0$

$\operatorname{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$

Well defined:

Lemma: Let H, H' be abelian gps w/ free resolutions F, F'. Then any ~~two~~ homomorphism $\alpha : H \to H'$ extends to a chain map:

$$
\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r] \arrow[d, dashed, "\alpha_2"'] & F_1 \arrow[r] \arrow[d, dashed, "\alpha_1"] & F_0 \arrow[r] \arrow[d, dashed, "\alpha_0"] & H \arrow[r] \arrow[d, "\alpha"] & 0 \\
\cdots \arrow[r] & F_2' \arrow[r] & F_1' \arrow[r] & F_0' \arrow[r] & H' \arrow[r] & 0
\end{tikzcd}
$$

and any two such extensions are chain homotopic.

Proof: Follow your nose diagram chase.

$$
\begin{tikzcd}
F_{n+1} \arrow[r] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] & F_n \arrow[r] \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \arrow[dl, dashed, "h"] & F_{n-1} \arrow[d, shift right, "\alpha"'] \arrow[d, shift left, "\beta"] \arrow[dl, dashed, "h"] \\
F_{n+1}' \arrow[r] & F_n' \arrow[r] & F_{n-1}'
\end{tikzcd}
$$

$\partial \circ h + h \circ \partial = \alpha - \beta$

Chain homotopy.

Lemma For any two free resolutions F and F' of H, there are canonical isomorphisms $H^n(F;G) \cong H^n(F';G)$

Cor: $\mathrm{Ext}(H,G) \cong H^1(F;G)$ is well defined

Pf of 2nd Lemma:

From:
$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\alpha"] \arrow[dd, bend right=60, "id"'] & H \arrow[d, "id"] \\
F' \arrow[r] \arrow[d, "\beta"] & H \arrow[d, "id"] \\
F \arrow[r] & H
\end{tikzcd}$$

$\beta \circ \alpha$ is chain homotopic to id.

Get:
$$\begin{tikzcd}
H^n(F;G) \\
H^n(F';G) \arrow[u, "\alpha^* \text{ indep of } \alpha"'] \\
H^n(F;G) \arrow[u, "\beta^* \text{ indep of } \beta"']
\end{tikzcd}$$

Thus: $\alpha^* \circ \beta^* = 1_{H^n(F;G)}$ and $\beta^* \circ \alpha^* = 1_{H^n(F';G)}$

$\Rightarrow \alpha^*, \beta^*$ are isomorphisms, canonical by Lemma 1. $\blacksquare$

Proof of the U.C.T. $0 \to \ker h \to H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n, G) \to 0$

Short exact seq of chain complexes

$$\begin{tikzcd}
 & \downarrow & \downarrow & \downarrow & \\
0 \arrow[r] & Z_{n+1} \arrow[r] \arrow[d, "0"'] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "\partial"'] & B_n \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, "i"] \arrow[d] & C_n \arrow[r] \arrow[d] \arrow[l, dashed, bend left] & B_{n-1} \arrow[r] \arrow[d] \arrow[l, dashed, bend left, "\partial"] & 0 \\
 & \, & \, & \, &
\end{tikzcd}$$

each row is split exact.

Apply $\mathrm{Hom}(-, G)$

$\varphi_1 \circ \partial \mapsfrom \varphi_0|_{B_n}$

$$\begin{tikzcd}
 & \uparrow & \uparrow & \uparrow & \\
0 & Z^*_{n+1} \arrow[l] & C^*_{n+1} \arrow[l] \arrow[u] & B^*_n \arrow[l, "\delta"'] \arrow[u] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l, "i^*"'] \arrow[u, "\delta"] & B^*_{n-1} \arrow[l, "\delta"'] \arrow[u, "0"] & 0 \arrow[l] \\
 & \uparrow & \uparrow & \uparrow &
\end{tikzcd}$$

$\varphi_0 \mapsfrom \varphi_1$

each row is split exact. (see next page)

Take the long exact sequence

$$\leftarrow B^*_n \xleftarrow{i^*_n} Z^*_n \leftarrow H^n(C;G) \xleftarrow{\delta} B^*_{n-1} \xleftarrow{i^*_{n-1}} Z^*_{n-1} \leftarrow$$

The connecting homomorphism is just i^*_n where $i_n : B_n \to Z_n$ is inclusion.

Note: $\mathrm{Hom}(\_, G)$ is left-exact, i.e. given an exact

(*) $0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$

get exact

$$\mathrm{Hom}(A,G) \leftarrow \mathrm{Hom}(C,G) \leftarrow \mathrm{Hom}(B,G) \leftarrow 0$$

When (*) splits, get

$$\mathrm{Hom}(C,G) \cong \mathrm{Hom}(A,G) \oplus \mathrm{Hom}(B,G)$$

Hence

$$\begin{array}{ccccccccc} 0 & \leftarrow & \ker i_n^* & \xleftarrow{h} & H^n(C,G) & \leftarrow & \mathrm{coker}\, i_{n-1}^* & \leftarrow & 0 \\ & & \parallel & & & & \parallel & & \\ & & \mathrm{Hom}(H_n, G) & & & & \mathrm{Ext}(H_{n-1}, G) & & \end{array}$$

Point: $0 \to \underbrace{B_{n-1} \xrightarrow[i_{n-1}]{} Z_{n-1}}_{\text{free}} \longrightarrow H_{n-1} \to 0$

$$0 \leftarrow B_{n-1}^* \xleftarrow[i_{n-1}^*]{} Z_{n-1}^* \leftarrow H_{n-1}^* \leftarrow 0$$

$$\mathrm{Ext}(H_{n-1}, G) = \mathrm{coker}\, i_{n-1}^*$$