

# Lecture 3: Universal Coefficient Thm

(1)

$C = \{ \dots \rightarrow C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \rightarrow \dots \}$  chain complex of free abelian gps.

$H_n =$  Homology of  $C$

$H^n(C; G) =$  Cohomology of  $\text{Hom}(C_n, G)$   $\leftarrow C^{n+1} \xleftarrow{\delta} C^n \leftarrow C^{n-1}$

Universal Coeff Thm: The following is exact:

$$0 \rightarrow \text{Ext}(H_{n-1}, G) \rightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n, G) \rightarrow 0$$

and moreover splits.



Start with  $h: [ \varphi ] \in H^n(C; G)$   $\varphi \in C^n = \text{Hom}(C_n, G)$   
 $[ \alpha ] \in H_n$   $\alpha \in C_n$

$$\boxed{\delta \varphi = 0, \partial \alpha = 0}$$

~~Set  $h([\varphi])([\alpha]) = \varphi(\alpha) \in G$~~

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Well defined: (a) Replacing  $\alpha$  with  $\alpha + \partial \beta$

$$\varphi(\alpha + \partial \beta) = \varphi(\alpha) + \varphi(\partial \beta) = \varphi(\alpha) + \delta \varphi(\beta) = \varphi(\alpha). \checkmark$$

(b) Replacing  $\varphi$  with  $\varphi + \delta\psi$  is OK since

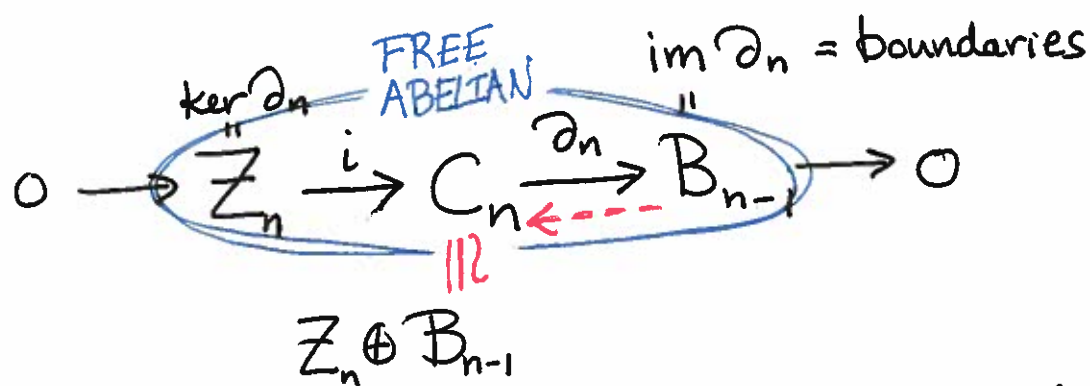
(2)

$$\delta\psi(\alpha) = \psi(\partial\alpha) = \psi(0) = 0 \quad \checkmark$$



Lemma:  $h$  is onto.

Pf: Let  $\varphi_0 \in \text{Hom}(H_n, G)$ . Consider



Get  $\rho: C_n \rightarrow Z_n$  where  $\rho \circ i = \text{id}|_{Z_n}$ . Define

$$\varphi_1: C_n \rightarrow G \text{ by } \begin{array}{ccccc} C_n & \xrightarrow{\rho} & Z_n & \rightarrow & H_n & \rightarrow & G \\ & & & & \searrow & \nearrow & \\ & & & & & \varphi_1 & \end{array}$$

$\in C^n(C; G)$

$$\text{Now } \delta\varphi_1(\alpha) = \varphi_1(\partial\alpha) = \varphi_0([\partial\alpha]) = 0$$

$\uparrow \in C_{n+1}$

So  $\varphi_1 \in C^n(C; G)$  is a cocycle where  $h(\varphi_1) = \varphi_0$ .

So  $h$  is onto.



Ext: The derived functor of  $\text{Hom}(-, G)$

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Def: A free resolution of an abelian gp  $H$  is an exact seq

$$\cdots \rightarrow F_3 \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow H \rightarrow 0$$

where each  $F_i$  is free abelian.

Ex:  $H = \mathbb{Z} \oplus \mathbb{Z}/2$

$$\textcircled{a} \quad 0 \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{1 \mapsto (0,2)} \mathbb{Z}^2 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow 0$$

$$\textcircled{b} \quad \cdots \rightarrow \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \mathbb{Z}^3 \rightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \rightarrow 0$$

Def:  $\text{Ext}(H, G)$  is the first cohomology gp of

$$\leftarrow F_2^* \leftarrow F_1^* \leftarrow F_0^* \leftarrow H^* \leftarrow 0$$

where  $A^* = \text{Hom}(A, G)$  and  $F_i$  are any free resolution of  $H$ .

Easy rules:  $\text{Ext}(H \oplus H', G) = \text{Ext}(H, G) \oplus \text{Ext}(H', G)$  (4)

If  $H$  is free  $\text{Ext}(H, G) = 0$   $[0 \rightarrow 0 \rightarrow H \xrightarrow{\cong} H \rightarrow 0]$

$\text{Ext}(\mathbb{Z}/n, G) = G/nG$ .  $[0 \rightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \rightarrow \mathbb{Z}/n \rightarrow 0]$

$\text{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2$      $\text{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/3) = 0$

$\text{Ext}(\mathbb{Z} \oplus \mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$

Well defined:

Lemma: Let  $H, H'$  be abelian gps w/ free resolutions  $F, F'$

Then any ~~map~~ homomorphism  $\alpha: H \rightarrow H'$  extends to a chain map:

$$\begin{array}{ccccccc} \rightarrow & F_2 & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow H \rightarrow 0 \\ & \downarrow d_2 & & \downarrow d_1 & & \downarrow d_0 & \downarrow \alpha \\ \dots & \rightarrow & F_2' & \rightarrow & F_1' & \rightarrow & F_0' \rightarrow H' \rightarrow 0 \end{array}$$

and any two such extensions are chain homotopic.

Proof: Follow your nose diagram chase.

$$\begin{array}{cccc} F_{n+1} & \rightarrow & F_n & \rightarrow & F_{n-1} \\ \alpha \downarrow \downarrow \beta & \swarrow h & \downarrow \alpha \downarrow \beta & \swarrow h & \downarrow \alpha \downarrow \beta \\ F_{n+1}' & \rightarrow & F_n' & \rightarrow & F_{n-1}' \end{array}$$

Chain homotopy.

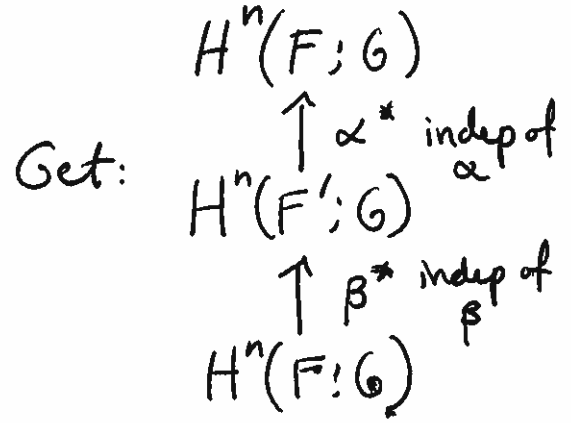
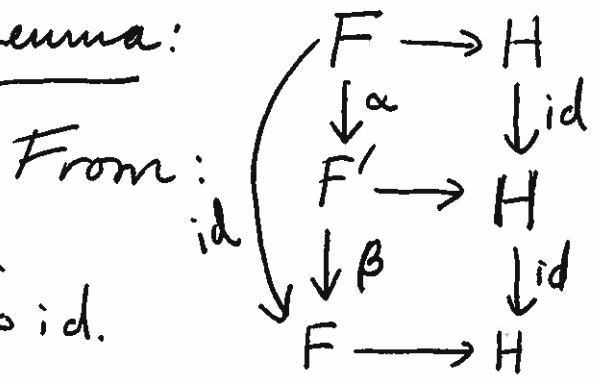
$\partial h + h \partial = \alpha - \beta$

Lemma For any two free resolutions  $F$  and  $F'$  of  $H$ , there are canonical ~~maps~~ isomorphisms  $H^n(F; G) \cong H^n(F'; G)$

Cor:  $\text{Ext}(H, G) \cong H^1(F; G)$  is well defined

Pf of 2<sup>nd</sup> Lemma:

$\beta \circ \alpha$  is chain homotopic to  $\text{id}$ .



Thus:  $\alpha^* \circ \beta^* = 1_{H^n(F; G)}$  and  $\beta^* \circ \alpha^* = 1_{H^n(F'; G)}$

$\Rightarrow \alpha^*, \beta^*$  are isomorphisms, canonical by Lemma 1.  $\square$

Proof of the U.C.T.  $0 \rightarrow \ker h \rightarrow H^n(C; G) \xrightarrow{h} \text{Hom}(H_n, G) \rightarrow \dots$

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Short exact seq  
of chain complexes

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & Z_{n+1} & \rightarrow & C_{n+1} & \xrightarrow{\partial} & B_n \rightarrow 0 \\
 & & \downarrow 0 & \swarrow \partial & \downarrow \partial & \swarrow \partial & \downarrow 0 \\
 0 & \rightarrow & Z_n & \xrightarrow{i} & C_n & \xrightarrow{\partial} & B_{n-1} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow
 \end{array}$$

each row  
is  
split  
exact.

Apply  $\text{Hom}(-, G)$

$$\begin{array}{ccccccc}
 & & \uparrow \varphi_1 & \leftarrow \delta & \uparrow \varphi_0 & & \\
 0 & \leftarrow & Z_{n+1}^* & \leftarrow & C_{n+1}^* & \leftarrow & B_n^* \leftarrow 0 \\
 & & \uparrow 0 & & \uparrow \delta & & \uparrow 0 \\
 0 & \leftarrow & Z_n^* & \xleftarrow{i^*} & C_n^* & \xleftarrow{\delta} & B_{n-1}^* \leftarrow 0 \\
 & & \uparrow \varphi_0 & \leftarrow \varphi_1 & \uparrow & & \uparrow
 \end{array}$$

each row  
is split  
exact.  
(see next  
page)

Take the long exact sequence

$$\leftarrow B_n^* \xleftarrow{i_n^*} Z_n^* \leftarrow H^n(C; G) \xleftarrow{\delta} B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow \dots$$

The connecting homomorphism is just  $i_n^*$  where  $i_n: B_n \rightarrow Z_n$  is inclusion.

Note:  $\text{Hom}(-, G)$  is left-exact, i.e. given an exact

$$\textcircled{*} \quad 0 \rightarrow A \rightarrow C \rightarrow B \rightarrow 0$$

get exact

$$\text{Hom}(A, G) \leftarrow \text{Hom}(C, G) \leftarrow \text{Hom}(B, G) \leftarrow 0$$

When  $\textcircled{*}$  splits, get

$$\text{Hom}(C, G) \cong \text{Hom}(A, G) \oplus \text{Hom}(B, G)$$

Hence

$$\begin{array}{ccccccc} 0 & \leftarrow & \text{Ker } i_n^* & \xleftarrow{h} & H^n(C, G) & \leftarrow & \text{coker } i_{n-1}^* \leftarrow 0 \\ & & \parallel & & & & \parallel \\ & & \text{Hom}(H_n, G) & & & & \text{Ext}(H_{n-1}, G) \end{array}$$

Point:  $0 \rightarrow \overbrace{B_{n-1}}^{\text{free}} \xrightarrow{i_{n-1}} Z_{n-1} \rightarrow H_{n-1} \rightarrow 0$

$$0 \leftarrow B_{n-1}^* \xleftarrow{i_{n-1}^*} Z_{n-1}^* \leftarrow H_{n-1}^* \leftarrow 0$$

$$\text{Ext}(H_{n-1}, G) = \text{coker } i_{n-1}^*$$