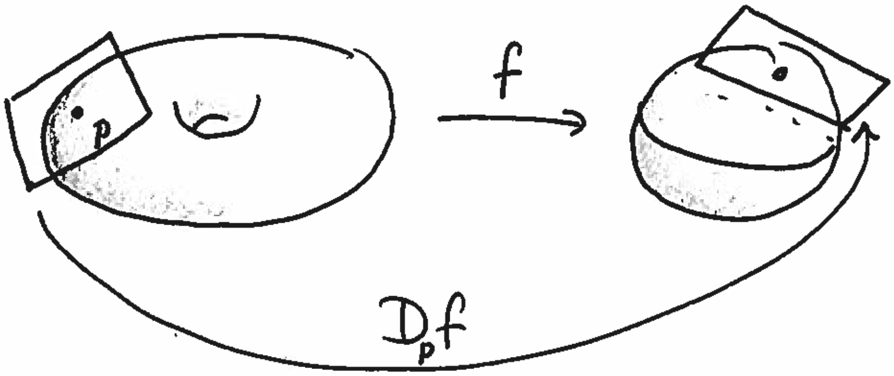
Lecture 4: Tangent spaces.

Goal: For a smooth $f: M \to N$ of smooth mflds define vector spaces $T_p M$ and $T_{f(p)} N$ with a



linear transformation $D_p f$ approximating f near p.

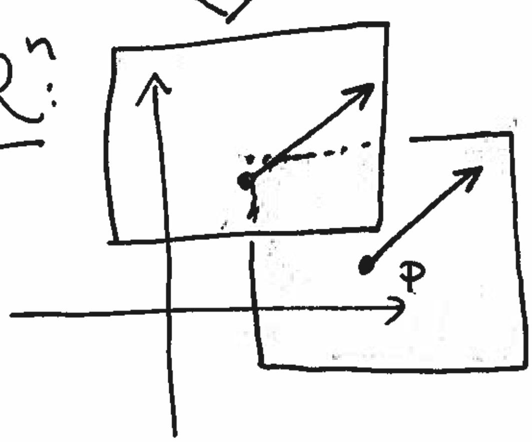
Motivating example: $S^2 = \{ p \in \mathbb{R}^3 \mid |p| = 1 \}$

$T_p S^2 =$ plane tangent to S^2 at p $= \{ x \in \mathbb{R}^3 \mid (x-p)\cdot p = 0 \iff x \cdot p = 1 \}$

$\overset{\text{kinda}}{\underset{\text{but not really}}{=}} \{ x \cdot p = 0 \}$

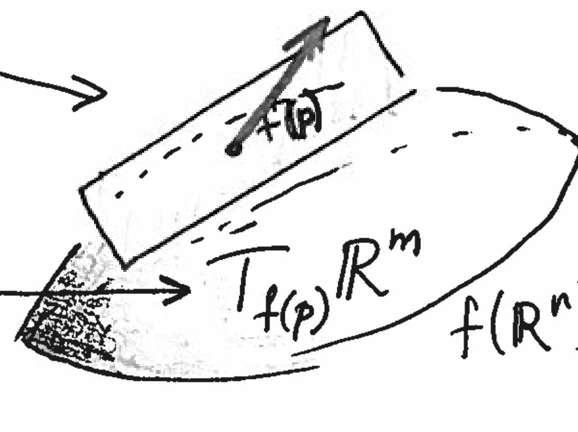
[In Math 241, these are equal. No MORE!]

$\mathbb{R}^n$:



$T_p \mathbb{R}^n = \{p\} \times \mathbb{R}^n = \{ (p, v) \mid v \in \mathbb{R}^n \}$.

$\mathbb{R}^m$



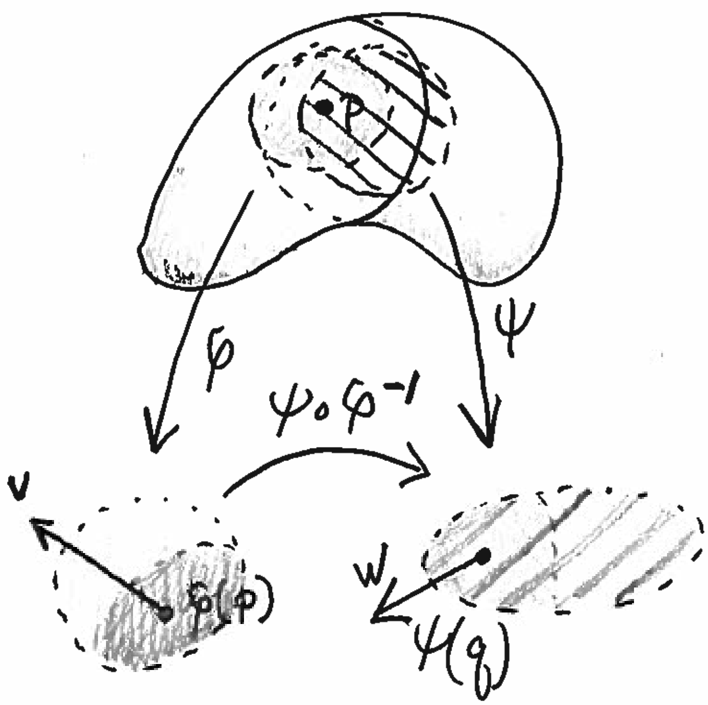
$f: \mathbb{R}^n \to \mathbb{R}^m$ smooth

$$D_p f = \left[\frac{\partial f_i}{\partial x_j} \right] : T_p \mathbb{R}^n \to T_{f(p)} \mathbb{R}^m$$

Note this still makes sense:

$$\underset{\text{pt } \mathbb{R}^n}{f(p} + \underset{T_p\mathbb{R}^n}{v}) = \underset{\mathbb{R}^m}{f(p)} + \underbrace{D_p f(v)}_{\in T_{f(p)}\mathbb{R}^m} + \underbrace{\mathrm{Error}_p(v)}_{O(|v|^2)}$$

Tangent vectors to a smooth M: (p, U, φ, v) where (U, φ) is a smooth chart with $p \in U$ and $v \in T_{\varphi(p)}\mathbb{R}^n$, up to the following equivalence: $(p, U, \varphi, v) \sim (q, V, \psi, w)$ if $p = q$ and $D_{\varphi(p)}(\psi \circ \varphi^{-1})(v) = w$.



$T_p M =$ set of such (p, U, φ, v)

Note we can add such

$$(p, U, \varphi, v_1) + (p, U, \varphi, v_2) = (p, U, \varphi, v_1 + v_2)$$

at least when (U, φ) match.

Makes sense beyond this since $D_{\varphi(p)}(\psi \circ \varphi^{-1})$ is a linear transformation.

Prop: For $(\overset{p}{\wedge}\!\!\!U, \varphi)$ the map $T_{\varphi(p)}\mathbb{R}^n \to T_p M$

$v \longmapsto (p, \varphi, U, v)$

is an isomorphism of vector spaces.

[Pf. Basically like #4 on HW.] [But there's another way...]

[Query: How many different ways can you think of a deriv. fn of 1-var?]

Back to $\mathbb{R}^n$: f smooth $\mathbb{R}^n \to \mathbb{R}$

$v \in T_a\mathbb{R}^n$

Directional derivative of f along v at a $= \dfrac{d}{dt} f(a+tv)\Big|_{t=0}$

$= (\nabla f(a)) \cdot v$

$= (D_a f)(v)$ [Ex: $\partial/\partial x_i$]

Set $C^\infty(\mathbb{R}^n) = \{\text{smooth } f: \mathbb{R}^n \to \mathbb{R}\}$ ← Vector space/$\mathbb{R}$

Have $D_{v_a}: C^\infty(\mathbb{R}^n) \longrightarrow \mathbb{R}$

$f \longmapsto (D_a f)(v)$

<u>Properties</u>: $\forall f, g \in C^\infty(\mathbb{R}^n)$ and $c \in \mathbb{R}$

a) $D_{v_a}(cf+g) = c\, D_{v_a} f + D_{v_a} g$

b) $D_{v_a}(f \cdot g) = (D_{v_a} f) \cdot g(a) + f(a)\,(D_{v_a} g)$

[Idea: a) and b) <u>Characterize</u> directional. der.]

A map $w: C^\infty(\mathbb{R}) \to \mathbb{R}$ is a <u>derivation at a</u> if

a) $w(cf+g) = c\, w(f) + w(g)$ ($\mathbb{R}$-linear)

b) $w(fg) = w(f)\, g(a) + f(a)\, w(g)$

for all $f, g \in C^\infty(\mathbb{R})$ and $c \in \mathbb{R}$.

<u>Prop</u>: $\mathcal{D}_a = \{\text{set of derivations at } a\}$

The map
$$T_a\mathbb{R}^n \longrightarrow \mathcal{D}_a$$
$$v_a \longmapsto D_{v_a}$$
is an isomorphism of $\mathbb{R}$-vector spaces.

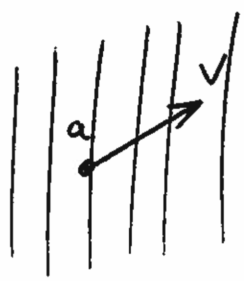
[Say why this will be help full]

Pf: That this is a linear transformation of $\mathbb{R}$-vector spaces follows from

$$D_{v_a + c w_a} f = D_{v_a} f + c D_{w_a} f$$

1-1: Suppose $v_a \longmapsto \left(\underset{0}{\text{const derivation}}\right)$. If

$\underbrace{v_i}_{i^{th}\text{ coor.}} \neq 0$ then $D_{v_a} x_i \neq 0$. So $v = 0$



$$\boxed{\text{Point: } D_{v_a} x_i = (\nabla x_i) \cdot v = e_i \cdot v = v_i}$$

onto: Let $w \in \mathcal{D}_a$. Set $v_i = w(x_i)$ and consider $v \in T_a \mathbb{R}^n$. Claim: $D_v = w$.

Suppose $f \in C^\infty(\mathbb{R}^n)$. Then

$$\frac{1}{2}\int_0^1 (1-t)^2 \frac{\partial}{\partial x_i x_j} f(a+t(x-$$

$$f(x) = f(a) + (D_a f)(x-a) + \text{Error}$$

$$= f(a) + \sum_i \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \sum_{ij} (x_i - a_i)(x_j - a_j) \underbrace{E_{ij}(x)}_{\text{smooth fn.}}$$

Now $w(1) = w(1\cdot 1) = w(1)\cdot 1 + 1\cdot w(1) = 2\,w(1)$

$\Rightarrow w(\text{const}) = 0$

If g & h both vanish at a, then

$w(g\cdot h) = 0.$

So $w(f) = 0 + \sum \frac{\partial f}{\partial x_i}(a)\big(w(x_i) - 0\big) + 0$

$= \sum \frac{\partial f}{\partial x_i}(a)\, v_i = \nabla f(a)\cdot v = D_{v_a} f.$

$\blacksquare$

So $f \longmapsto w$ as needed