Lecture 15: Lie groups: subgps and actions.

Def: G, H Lie groups. A $F: G \to H$ is a Lie group homomorphism if it is a group homomorphism and a smooth map.

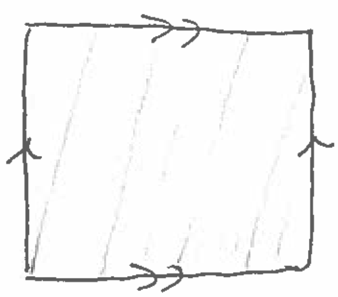
Ex: $\pi: \mathbb{R}^2 \longrightarrow T^2 = S^1 \times S^1$ where $\pi(s,t) = (e^{2\pi i s}, e^{2\pi i t})$

Lie subgp: Image $F(H)$ of an injective Lie gp homomorphism $F: G \to H$.

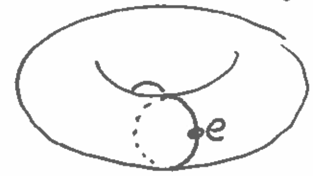
Note: Such F is an immersion [since F has const rank] and so $F(H)$ is an immersed submfld.

Ex: $H = \mathbb{R}$, $G = S^1 \times S^1$, $\alpha \in \mathbb{R}$ irrational

$$F(t) = \pi(t, \alpha t) = (e^{2\pi i t}, e^{2\pi i \alpha t})$$



$F(H)$ is not an embedded submanifold. Contrast if $\alpha = 0$



Thm: $H \subseteq G$ is a Lie subgroup. Then H is closed in G iff H is an embedded submfld.

Thm: Suppose G is a Lie group. If $A \subseteq G$ is a closed subset which is also a (non-Lie) subgroup of G, then A is an embedded Lie subgp of G.

Pfs: Lee 7.21 and 20.12.

A Lie group action of a Lie gp G on a smooth manifold M is a smooth
$$G \times M \longrightarrow M$$
$$(g, m) \longmapsto g \cdot m$$
where
$$\left.\begin{array}{l} g_1 \cdot (g_2 \cdot m) = (g_1 \cdot g_2) \cdot m \\ \text{and} \quad e \cdot m = e \end{array}\right\} \forall g_1, g_2 \in G \text{ and } m \in M$$

[These are just the def. of a left action of a gp on a set plus the assumption that the "action map" is smooth. Can also talk about right actions...]

Ex: (a) Trivial action: $g \cdot m = m$ for $\forall g \in G$ and $m \in M$.

(b) Linear action of $\underset{\substack{\cup\!\!\!\!\shortmid \\ A}}{GL_n \mathbb{R}}$ on $\underset{\substack{\cup\!\!\!\!\shortmid \\ v}}{\mathbb{R}^n}$: $A \cdot v = Av$ ← as column vector.

[Explain why sat the axioms.]

(c) $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \;\middle|\; \begin{matrix} a \in \mathbb{R}^\times \\ b \in \mathbb{R} \end{matrix} \right\}$ acts on $\mathbb{R}$ by

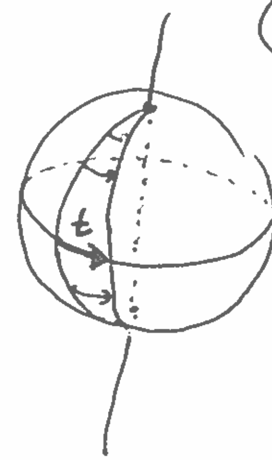
$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \cdot t = at + b \qquad \text{[Check!]}$$

Sometimes give action a name $\Theta : G \times M \longrightarrow M$ and write $g \cdot m = \Theta_g(m)$. Then $\Theta_g : M \longrightarrow M$ is $\Theta|_{\{g\} \times M}$, and we have

$$\Theta_{g_1} \circ \Theta_{g_2} = \Theta_{g_1 g_2}$$

Thus Θ_g is a diffeo of M with inverse $\Theta_{g^{-1}}$ since $\Theta_g \circ \Theta_{g^{-1}} = \Theta_e = \mathrm{id}_M = \Theta_e = \Theta_{g^{-1}} \circ \Theta_g$.

<u>Ex</u>: $G = \mathbb{R}$, $M = S^2 \subseteq \mathbb{R}^3$ where

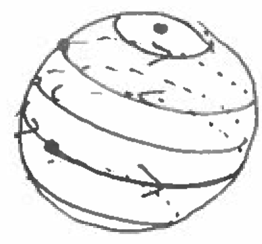


$$\Theta_t(m) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}$$

= rotation by angle t around the z-axis.

Check: $\Theta_{t_1} \circ \Theta_{t_2} = \Theta_{t_1 + t_2}$.

<u>Orbit</u>: $G \cdot m = \{ g \cdot m \mid g \in G \}$



[M is the dijoint union of the various orbits]

<u>Stabilizer</u>/<u>Isotropy group</u>: $G_m = \{ g \in M \mid g \cdot m = m \}$

$G_{\substack{\text{pt on}\\ \text{equator}}} = 2\pi\mathbb{Z}$ $\qquad G_{\text{north pole}} = \mathbb{R}$

<u>Orbit Stabilizer Theorem</u>: $G \cdot m = G / G_m$

[For now, haven't said how to put a smooth structure on G/G_m. Follows from some variant of the const rank thm for Lie grp homomorphisms]

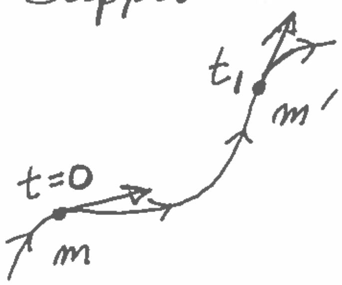
$\mathbb{R}$ actions, flows, and vector fields:

Suppose Θ is an action of $\mathbb{R}$ on M. Set

$\Theta^{(m)} : \mathbb{R} \longrightarrow M$ to be the path $\Theta^{(m)}(t) = \Theta_t(m)$

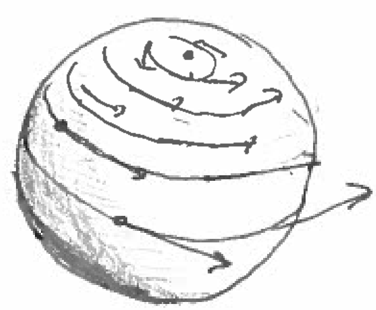
[so $\Theta^{(m)}(\mathbb{R}) = \mathbb{R} \cdot m$]

Suppose $m' = \Theta^{(m)}(t_1)$. Note that



$$\Theta^{(m')}(t) = \Theta_t(m') = \Theta_t(\Theta_{t_1}(m))$$
$$= \Theta_{t+t_1}(m) = \Theta^{(m)}(t+t_1)$$

The *infinitessimal generator* $V \in \mathfrak{X}(M)$ of Θ is defined to be



$$V_m = \frac{d}{dt}\Theta^{(m)}\Big|_{t=0}$$
$$= d\Theta\left(\frac{\partial}{\partial t}\Big|_{(0,m)}\right)$$

[From the second description, V is clearly smooth.]

Note that $\forall t \in \mathbb{R}$, $m \in M$ we have

$$\left(\theta^{(m)}\right)'(t) = \left(\theta^{(m')}\right)'(0) = V_{m'} = V_{\theta^{(m)}(t)}.$$

where $m' = \theta^{(m)}(t)$

That is, $\theta^{(m)}$ is the integral curve

for V with position m at time 0.

Next time, will (mostly) reverse this?