Lecture 9: Inverse Function Thm

Previously: Immersions, submersions, covering maps.

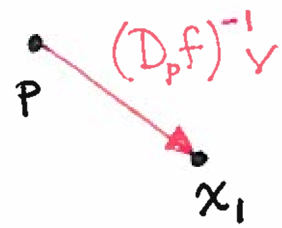
IFT: Suppose $F : (U \underset{\text{open}}{\subseteq} \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is smooth. If D_pF is invertible, then $\exists$ an open ball B about p such that

$$\left.\begin{array}{l} \text{① } F|_B \text{ is 1-1} \\ \text{② } F(B) \text{ is open} \\ \text{③ } (F|_B)^{-1} \text{ is smooth} \end{array}\right\} \quad F|_B \text{ is a diffeomorphism between open subsets of } \mathbb{R}^n.$$

[Motivation: derivative as linear approximation.]

Computing the inverse: Given y, ← NEAR $F(p)$

Seek x with $f(x) = y$



First guess: $x_0 = p$

$$\left.\begin{array}{l} \text{Second guess: } x_1 = x_0 + (D_pf)^{-1}(y - f(x_0)) \\ \qquad\qquad\quad\; x_2 = x_1 + (D_{x_1}f)^{-1}(y - f(x_1)) \\ \qquad\qquad\qquad \vdots \end{array}\right\} \text{Newton's Method}$$

Replace with D_pf

Define: $\Phi_y(x) = x + (D_pf)^{-1}(y - f(x))$

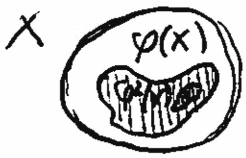
$$x_{n+1} = \Phi_y(x_n)$$

Note: $\Phi_y(x) = x \iff f(x) = y$.

[General technique: Replace finding a solution to some equations with a fixed-point problem.]

<u>Contractions</u>: $\varphi : X \to X$

$\exists C < 1$ where $d(\varphi(x_1), \varphi(x_2)) \leq C\, d(x_1, x_2)$



<u>Contraction Mapping Thm</u>: Any contraction of a <u>complete</u> metric space has a unique fixed pt.

<u>Sketch proof</u>: $x_0 \in X$ some pt. Set $x_{n+1} = \varphi(x_n)$.

Then $d(x_n, x_{n+1}) \leq C\, d(x_{n-1}, x_n) \leq C^n d(x_0, x_1)$.

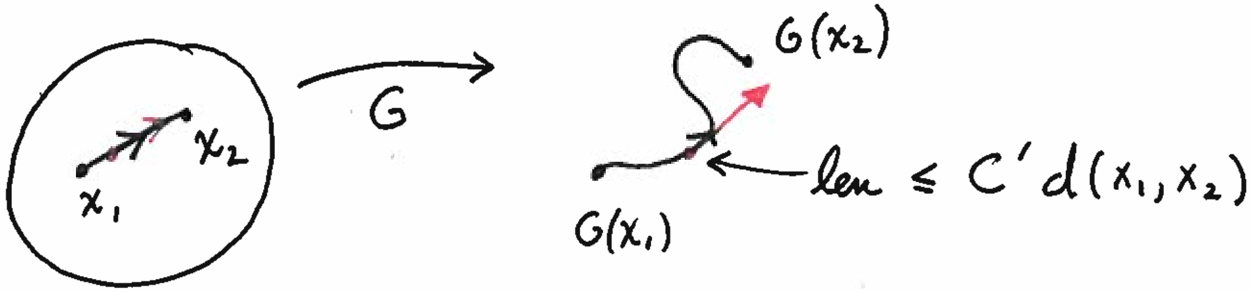
This Cauchy sequence converges to ~~a~~ the fixed pt. ▨

<u>Def</u>: $A : \mathbb{R}^n \to \mathbb{R}^n$ linear. The norm of A is

$$|A| = \sup_{v \in \mathbb{R}^n \setminus \{0\}} \frac{|Av|}{|v|}$$

[Max. anything is stretched.]

Suppose $B \subseteq \mathbb{R}^n$ is a ball, $G : B \to \mathbb{R}^n$ smooth and $\forall p \in B$ the derivative D_pG has $|D_pG| < C'$ for some fixed $C' < 1$. Then G is a contraction.



Proof of IFT ① Fix y. Then

$$D_x \Phi_y = I + 0 - (D_p f)^{-1}(D_x f)$$

Since $D_x f$ is a smooth fn of x, there is an $r_0 > 0$ so that on $B = B_{r_0}(p)$ we have is $\approx I$ and hence

$$|D_x \Phi_y| \le \frac{1}{2}$$ for $x \in \overline{B}$ and for any y.

Suppose $x_1, x_2 \in B$ with $F(x_1) = F(x_2)$.
Then x_1, x_2 are both fixed pts of the contraction $\Phi_{F(x_1)}\big|_B$ and hence $=$.

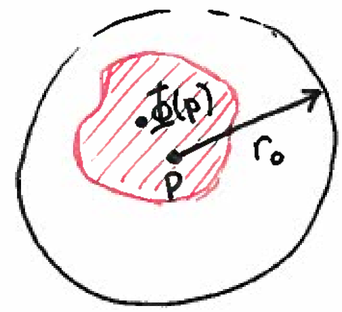
Proof of IFT ② It suffices to show
F is onto a small open ball about $F(p)$.

Choose ε so that

$$|(D_p f)^{-1}(y - f(p))| < \frac{1}{4} r_0 \text{ for all } y \in B_\varepsilon(f(p)).$$

Then fix $y \in B_\varepsilon(f(p))$. So

$$d(p, \Phi_y(p)) < \frac{1}{4} r_0$$



and $\Phi_y(\overline{B}) \subseteq \overline{B}$ since

if $x \in \overline{B}$ we have

$$d(\Phi_y(x), p) = d(\Phi_y(x), \Phi_y(p)) + d(\Phi_y(p), p)$$
$$\leq \frac{1}{2} d(x, p) + \frac{1}{4} r_0 \leq \frac{3}{4} r_0$$

So $\Phi_y|_{\overline{B}}$ is a contraction of a complete metric space,

and hence has a fixed pt. So $F(B) \supseteq B_\varepsilon(f(p))$.

<u>Pf of ③:</u> Will show that $(F|_B)^{-1}$ ~~has~~ is differentiable at p ~~on B~~; arguement for smoothness on B is similar.

Translating coordinates, can assume $p = 0 = F(p)$.

Since F is smooth, $\exists C, C'$ so that

$$|F(x) - (D_0 F)x| < C|x|^2 \quad \text{for } x \text{ near } 0.$$

and so that $\underbrace{|x| < C'|F(x)|}_{\text{e.g. } C' = |(D_0F)^{-1}| + 1}$ again for x near 0.

Then $\qquad$ Take x so that $y = F(x)$

$$\left| F^{-1}(y) - \underbrace{(D_0F)^{-1}y}_{\text{guess for derivative}} \right| = \left| (D_0F)^{-1}\left(y - (D_0F)\, F^{-1}(y) \right) \right|$$

$$\leq \left| (D_0F)^{-1} \right| \left| F(x) - (D_0F)(x) \right|$$

$$\leq \left| (D_0F)^{-1} \right| \cdot C \cdot (C')^2 \, |y|^2$$

Hence F^{-1} is diff at 0 with derivative $(D_0F)^{-1}$. $\blacksquare$