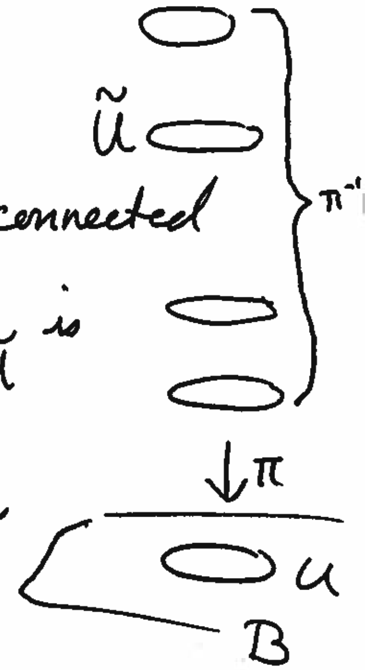
Lecture 8: Covering maps and submersions

Last time:

Immersion: Smooth $F: M \to N$ where $\forall p \in M$ the map $dF_p : T_pM \to T_{F(p)}N$ is 1-1.

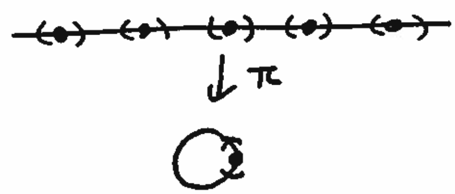
Suppose $\pi: E \to B$ is ~~a~~ smooth.

An open $U \subseteq B$ is evenly covered if $\forall$ connected components $\tilde{U}$ of $\pi^{-1}(U)$, then restriction $\pi|_{\tilde{U}}$ is a diffeomorphism.

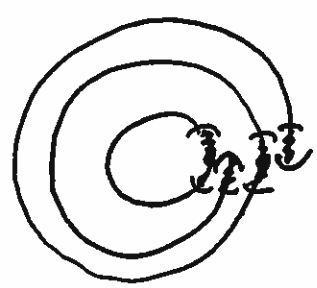


Smooth covering map: A smooth onto immersion $\pi: E \to B$ where every $b \in B$ is contained in an evenly covered nbhd.

Ex: $\pi: \mathbb{R} \to S$
$t \mapsto e^{it}$

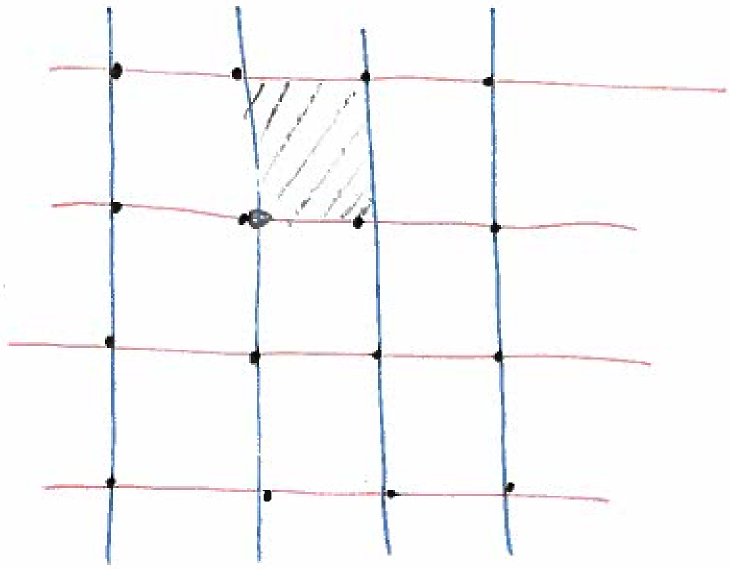


Non-ex: $\pi|_{(-2\pi, 2\pi)}$



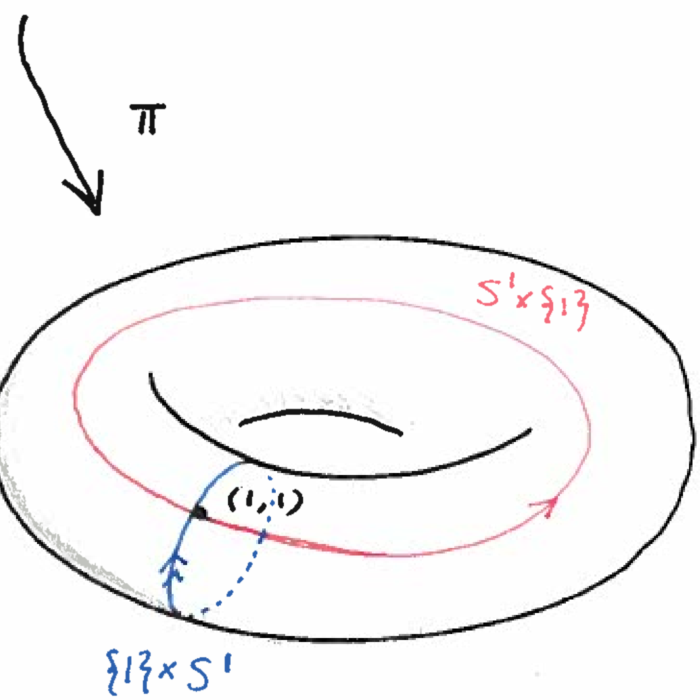
Ex: $\pi : \mathbb{R}^2 \longrightarrow T = S^1 \times S^1 \subseteq \mathbb{C}^2$

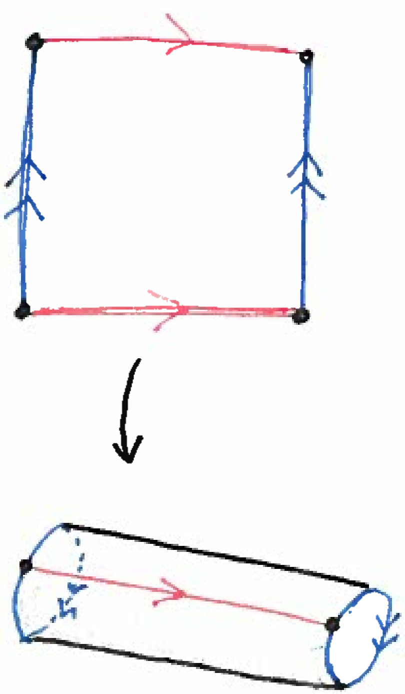
$$(s,t) \longmapsto \left(e^{2\pi i s}, e^{2\pi i t}\right)$$

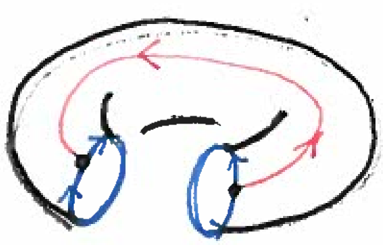


$\pi^{-1}(1,1) = \mathbb{Z}^2$

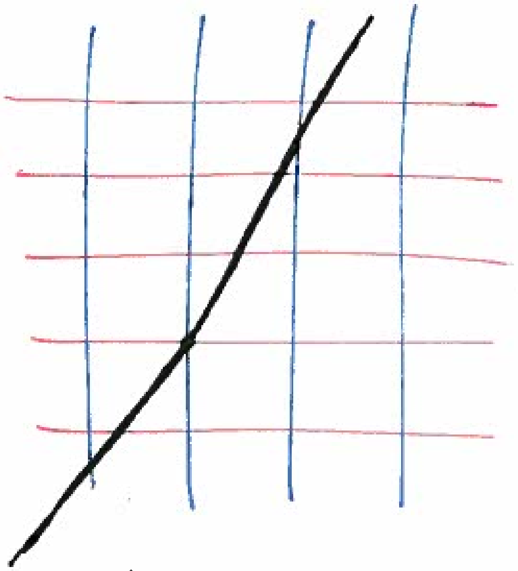
Set $D = [0,1] \times [0,1] \subseteq \mathbb{R}^2$.
Then map $\pi|_D$ is onto T
and 1-1 except on ∂D.





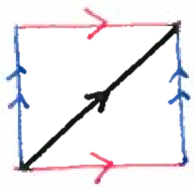


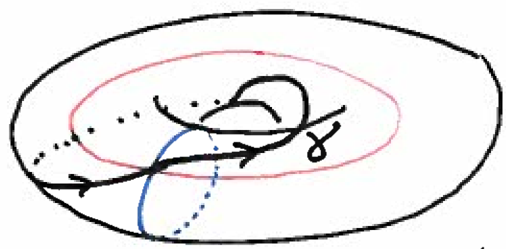
Fix $\alpha \in \mathbb{R}$. Define $\gamma(r) = \pi(r, \alpha r)$ to get a smooth immersion $\mathbb{R} \to T^2$.



line of slope α

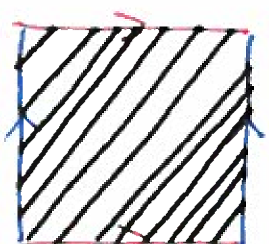
Ex: $\alpha = 1$





Not 1-1, but image is an embedded submfld.

Ex: $\alpha = \sqrt{2}$



Get a 1-1 smooth immersion of $\mathbb{R}$ in T with dense image! [HW.]

Alternate viewpt(s): ① $T^2 = \mathbb{R}^2 / (x_1, y_1) \sim (x_2, y_2)$ if $x_1 - x_2 \in \mathbb{Z}$, $y_1 - y_2 \in \mathbb{Z}$

[Thinking of Torus as a quotient]

② $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(a,b)\cdot(x,y) = (x+a, y+b)$

T^2 is the space of orbits under this gp action.

③ $\mathbb{R}^2$ is a gp under addition of vectors (Lie group!)

$\mathbb{Z}^2$ is a normal subgp. T^2 is the quotient group.

(Compare $\mathbb{R} \rhd \mathbb{Z}$ and $\mathbb{R}/\mathbb{Z} \cong S^1 \leftarrow$ group str as subgp of $\mathbb{C}^\times$)

In earlier example $\gamma(\mathbb{R}) =$ is a subgp of T^2.

Ex: $H = \{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$

$f(z) = z + 2$ and $g(z) = \dfrac{z}{2z+1}$ } Möbius trans, pres H

$$H/\langle f, g\rangle = S^2 \setminus \{3 \text{ pts}\}$$

$H \to H/\langle f,g\rangle$ a covering map.

$f^{-1}(z) = z - 2 \qquad g^{-1}(z) = \dfrac{z}{-2z+1}$

<u>Submerssion</u>: Smooth $F: M \to N$ so that $\forall p \in M$ the derivative $dF_p : T_p M \to T_{F(p)} N$ is <u>onto</u>.

<u>Ex</u>: (a) $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\qquad dF_p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$(x,y,z) \longmapsto (x,y)$

(b) $M \times N \xrightarrow{\pi} M$ $\quad$ [Apply HW on $T_{(m,n)} M \times N$]

$(m,n) \longmapsto m$

(c) $\mathbb{R}^3 \xrightarrow[F]{} \mathbb{R}$ $\qquad F(x,y,z) = x^2 + y^2 + z^2$

$dF = (2x, 2y, 2z) \leftarrow$ NOT SUBMERSION at $(0,0,0)$

~~So~~ Then $F|_{\mathbb{R}^3 \setminus \mathbb{R}}$ is a submersion.

<u>Non-ex</u>: (a) $\dim M < \dim N$ [Query about diffeos]

(b) $\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ $\qquad dF = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$(x,y,z) \longmapsto (x,x)$

<u>Note</u>: Many maps are not immersions or submersions.

Thm: If $M \xrightarrow{F} N$ is a submersion, then $\forall\, q \in N$ the preimage $F^{-1}(q)$ is a smooth embedded submanifold of M of dimension $\dim M - \dim N$.

Cor: $S^2 = F^{-1}(1)$ is a smooth mfld.

from © above.

[~~That is bec~~ Thm is because any submersion looks like a product in carefully choosen local coor.]

Inverse Function Thm: Suppose $F : (U \underset{\text{open}}{\subseteq} \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is smooth. If $\underset{\overset{\shortparallel}{D_pF}}{dF_p}$ is invertible at some $p \in U$, then $\exists$ an open ball ~~B~~ $B \subseteq U$ about p so that ~~$F|_B$ is a diffeomorphism from B to~~

(a) $F|_B$ is 1-1

(b) $F(B)$ is open

(c) $(F|_B)^{-1}$ is smooth

} F is a diffeo between the two open sets B and $F(B)$.