Lecture 27: k-forms and orientations

$$\Lambda^k(V) = \left\{ \begin{array}{c} \text{Antisymmetric } k\text{-tensors} \\ \alpha : \underbrace{V \times V \times \cdots \times V}_{k} \longrightarrow \mathbb{R} \end{array} \right\}$$

Wedge/Exterior product: $\alpha_1, \ldots, \alpha_k \in V^*$ define

$$\alpha_1 \wedge \cdots \wedge \alpha_k (v_1, v_2, \ldots, v_k) = \det \begin{pmatrix} \alpha_1(v_1) & \alpha_1(v_2) & \cdots & \alpha_1(v_k) \\ \alpha_2(v_1) & \ddots & & \vdots \\ \vdots & & & \\ \alpha_k(v_1) & \alpha_k(v_2) & \cdots & \alpha_k(v_k) \end{pmatrix}$$

Basis for $\Lambda^k(V)$:

$\{ \beta_{i_1} \wedge \cdots \wedge \beta_{i_k} \mid i_1 < \cdots < i_k \}$ where $\{\beta_i\}$ is a basis for V^*

Notes: ① $\Lambda^k(V) = 0$ if $k > \dim V$ since any k-tensor is determined by its values on tuples of basis elts and $\alpha \in \Lambda^k(V)$ is 0 on any tuple with a repeated vector.

② $\dim \Lambda^k(V) = \binom{\dim V}{k}$

③ $\alpha_{\sigma(1)} \wedge \alpha_{\sigma(2)} \wedge \cdots \wedge \alpha_{\sigma(k)} = (\operatorname{sgn} \sigma)\, \alpha_1 \wedge \cdots \wedge \alpha_k$

Also have $\Lambda^k(V) \times \Lambda^\ell(V) \xrightarrow[\wedge]{} \Lambda^{k+\ell}(V)$ which

is compatible with the one on 1-forms, i.e.

$$(\alpha_1 \wedge \cdots \wedge \alpha_k) \wedge (\beta_1 \wedge \cdots \wedge \beta_\ell) = (\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \beta_\ell)$$

and satisfies $\omega, \omega' \in \Lambda^k(V)$ $\quad \eta, \eta' \in \Lambda^\ell(V)$ $\quad a \in \mathbb{R}$

<u>Bilinear</u>: $(a\omega + \omega') \wedge \eta = a(\omega \wedge \eta) + \omega' \wedge \eta$
$\omega \wedge (a\eta + \eta') = a(\omega \wedge \eta) + \omega \wedge \eta'$

<u>Assoc</u>: $\zeta \in \Lambda^m(V)$ $\quad \omega \wedge (\eta \wedge \zeta) = (\omega \wedge \eta) \wedge \zeta$

<u>Anticommutative</u>: $\omega \wedge \eta = (-1)^{k\ell}\, \eta \wedge \omega$

<u>Actual definition</u>:

$$\omega \wedge \eta (v_1, v_2, \ldots, v_{k+\ell}) =$$

$$\frac{1}{k!\,\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \cdot \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})$$

[For details ponder and/or see Chap 14 of Lee or Chap 5 of Boothby]

Differential forms:

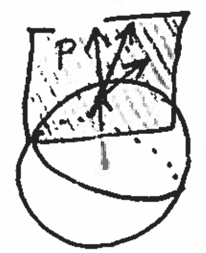
$$\Omega^k(M) = \left\{ \begin{array}{c} \text{smooth assignments} \\ p \longmapsto \Lambda^k(T_pM) \end{array} \right\}$$

Ex: $\omega_{(x,y)} = (x+y^2)\left(dx\big|_{(x,y)} \wedge dy\big|_{(x,y)}\right)$ on $\mathbb{R}^2$

$$\omega\left(\frac{\partial}{\partial x}\Big|_{(1,2)} + \frac{\partial}{\partial y}\Big|_{(1,2)},\ 3\frac{\partial}{\partial y}\Big|_{(1,2)}\right) = (1+2^2)\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix}$$

$$= 15$$

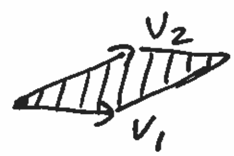
Ex: Define $\eta \in \Omega^2(S^2)$ by



$$\eta_p(v_1, v_2) = \det\begin{pmatrix} \text{---} & p & \text{---} \\ \text{---} & v_1 & \text{---} \\ \text{---} & v_2 & \text{---} \end{pmatrix}$$

where $p \in \mathbb{R}^3$ and $v_1, v_2 \in T_pS^2 \subseteq T_p\mathbb{R}^3 \cong \mathbb{R}^3$.

Note: This is the signed area of



[Just as with 1-forms...] For $F: M \to N$

get $F^*: \Omega^k(N) \to \Omega^k(M)$ via

$$(F^*\omega)_p(v_1, \dots, v_k) = \omega_{F(p)}(dF_p v_1, \dots, dF_p v_k)$$

Also, the wedge product on each $\Lambda^k(T_pM)$ gives rise to a wedge product $\Omega^k(M)$.

<u>Integration</u>: $\Lambda^n(T_p\mathbb{R}^n)$ is one dim'l with basis $dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. So $\omega \in \Omega^n(\mathbb{R}^n)$ can be written $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ where $f \in C^\infty(\mathbb{R}^n)$

Define for a bounded open set $M \subseteq \mathbb{R}^n$ the integral as :

$$\int_M \omega = \int \cdots \int_M f(x_1, \ldots, x_n)\, dx_1 dx_2 \cdots dx_n$$

<u>Q</u>: Is $\int_M \omega$ invariant under diffeo $F: M \to N \overset{\text{open}}{\subseteq} \mathbb{R}^n$?

<u>A</u>: As we'll see, the answer is yes <u>up to sign</u>.

<u>Q</u>: How do we define $\int_M \omega$ for a more complicated M?

[Combine with earlier example to compute the area(S^2).]

<u>Orientations</u>: An orientation on V ($\cong \mathbb{R}^n$) is either

ⓐ A choice of basis $\mathcal{B}$ for V where $\mathcal{B} \sim \mathcal{B}'$ if $\det [Id_V]_{\mathcal{B}'}^{\mathcal{B}} > 0$.

ⓑ A choice of connected component of $\underbrace{\Lambda^{\dim V}(V)}_{\mathbb{R}} \setminus \{0\}$.

<u>Correspondence</u>: $(\omega \neq 0 \text{ in } \Lambda^n V) \longrightarrow \begin{pmatrix} \text{Basis } e_1, \dots, e_n \\ \text{with } \omega(e_1, \dots, e_n) > 0 \end{pmatrix}$

<u>Proof</u>: For $\mathcal{B} = \{e_1, \dots, e_n\}$ let $\alpha_1, \dots, \alpha_n$ be the dual basis of V^*. Set $\omega = \alpha_1 \wedge \dots \wedge \alpha_n \neq 0$.

Then for $\mathcal{B}' = \{d_1, d_2, \dots, d_n\}$ you can check that

$$\omega(d_1, \dots, d_n) = \det [Id_V]_{\mathcal{B}'}^{\mathcal{B}}$$

as needed. $\blacksquare$