Lecture 26: Tensors and differential forms.

__Riemannian metric__: $p \longmapsto \left(g_p : T_pM \times T_pM \to \mathbb{R} \right)$ symmetric, pos def, bilinear form.

$$TM \xrightarrow{\cong} T^*M$$
$$v_p \longmapsto g_p(v_p, \cdot)$$

$$\mathfrak{X}(M) \longleftrightarrow \Omega^1(M)$$
$$X \longmapsto X^\flat$$
$$\omega^\# \longleftarrow\!\shortmid\ \omega$$

For $f \in C^\infty(M)$ set $\operatorname{grad} f = (df)^\#$

A bilinear form $g : V \times V \to \mathbb{R}$ is ~~may~~ __non-degenerate__ if $\forall x \overset{\ne 0}{\in} V$ $\exists y \in V$ with $g(x,y) \neq 0$.

__Ex__: g pos. definite.

__Prop__: If g is non-degen and V finite dim'l then $V \longmapsto V^*$, $v \longmapsto g(v, \cdot)$ is an isomorphism.

__Pf__: The map $V \to V^*$ is a linear trans by bilinearity of g. Non degen is exactly that this map has no kernel. Since $\dim V = \dim V^*$ we're done. ▨

For $f: \mathbb{R}^n \to \mathbb{R}$ the form $df \in \Omega^1(\mathbb{R}^n)$ is given by

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \qquad \left[\begin{array}{l} \text{Since } df(V) = Vf \\ \quad = D_V f = (\operatorname{grad} f)\cdot V \end{array}\right]$$

which is

$$\left(\operatorname{grad} f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i} \right)^{\flat}$$

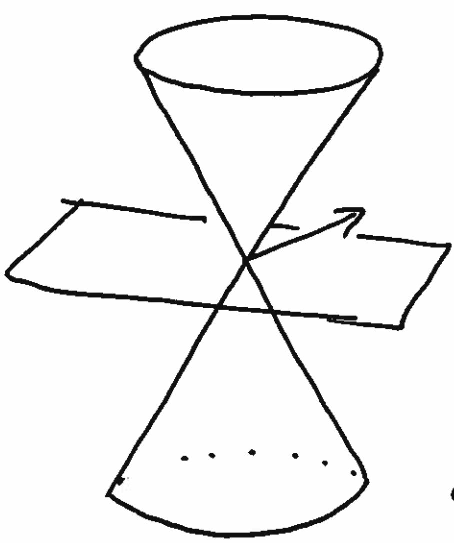
since g_{DOT} induces the isom $T_p\mathbb{R}^n \longleftrightarrow T_p^*\mathbb{R}^n$

$$\frac{\partial}{\partial x_i} \longleftrightarrow dx_i .$$

<u>Pseudo-Riemannian</u>: g_p nondegen. bilinear form on T_pM.

<u>Lorentzian</u>: $g((x_1, x_2, x_3), (y_1, y_2, y_3))$

$$= x_1 y_1 + x_2 y_2 - x_3 y_3$$



$g((1,0,1), (1,0,1)) = 0$

$g(x,x) = 0$ light-like

$g(x,x) > 0$ space-like

$g(x,x) < 0$ time like.

<u>Lorentzian Manifold</u>: A form of this type at each $p \in M$. [Basic object in general relativity.]

Multilinear form: $\underbrace{V \times V \times \cdots \times V}_{k} \xrightarrow[\alpha]{} \mathbb{R}$ is also called a "covariant k-tensor". Two special kinds:

<u>Symmetric</u>: $\alpha(v_1, v_2, \ldots, v_k) = \alpha(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(k)})$

for all $\sigma \in \mathrm{SymGp}(\{1, \ldots, k\}) = S_k$.

<u>Antisymmetric</u>: $\alpha(v_1, v_2, \ldots, v_k) = \operatorname{sgn}(\sigma)\, \alpha(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$

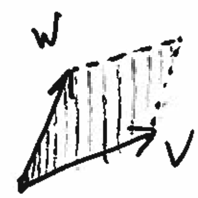
for all $\sigma \in S_k$.

[Alternatively, swapping any pair $v_i \leftrightarrow v_j$ changes α by -1.]

[Antisym also called k-forms, k-covectors, exterior forms...]

<u>Ex</u>: $V = \mathbb{R}^2$ $\quad \omega(v, w) = \det\begin{pmatrix} v_1 & v_2 \\ w_1 & w_2 \end{pmatrix}$

$= \text{signed area}$



<u>Ex</u>: $V = \mathbb{R}^n$ $\quad \omega(v_1, \ldots, v_n) = \det\begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{pmatrix}$

[= signed volume]

<u>Ex</u>: $\alpha, \beta \in V^*$ Define $\alpha \wedge \beta : V \times V \to \mathbb{R}$

by $\alpha \wedge \beta(v_1, v_2) = \big(\alpha(v_1)\beta(v_2) - \beta(v_1)\alpha(v_2)\big)$

$= \det\begin{pmatrix} \alpha(v_1) & \alpha(v_2) \\ \beta(v_1) & \beta(v_2) \end{pmatrix}$

$\Lambda^k(V)$: vector space of k-forms

For $\alpha_1, \dots, \alpha_k \in V^*$ define an elt of $\Lambda^k(V)$ by

$$\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k (v_1, v_2, \dots, v_k) = \det \begin{pmatrix} \text{---} \ \alpha_1(v_1) \ \text{---} \\ \text{---} \ \alpha_2(v_2) \ \text{---} \\ \vdots \\ \text{---} \ \alpha_k(v_k) \ \text{---} \end{pmatrix}$$

[Think of $(\alpha_1, \dots, \alpha_k)$ as defining a map $V \to \mathbb{R}^k$, taking volume of image of "parallelopided" in V with sides $v_1, \dots, v_k$.]

<u>Thm</u>: Suppose $\alpha_1, \dots, \alpha_n$ is a basis for V^*.
Then $\Lambda^k(V)$ has basis $\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} \mid i_1 < i_2 < \cdots < i_k\}$

<u>Ex</u>: $\mathbb{R}^3 = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ $\quad (\mathbb{R}^3)^* = \langle dx, dy, dz \rangle$

$\Lambda^1(V) = V^*$ basis dx, dy, dz

$\Lambda^2(V)$ has basis $dx \wedge dy \quad dx \wedge dz \quad dy \wedge dz$

$\Lambda^3(V)$ has basis $dx \wedge dy \wedge dz$

$\Lambda^k(V) = 0 \quad k > 3.$

<u>Note</u>: $\dim \Lambda^k(V) = \binom{n}{k}$

<u>Pf of Thm</u>: Basic idea. If $\{e_i\}$ is the dual basis for V, then by antisymm. of $\omega \in \Lambda^k(V)$, it is det by its values on $(e_{i_1}, e_{i_2}, \ldots, e_{i_k}) = e_I$ for $I = (i_1 < i_2 < \cdots < i_k) \in \mathcal{I}$. [Skip details.]

Set $\eta = \sum_{I \in \mathcal{I}} \omega(e_I)\, \alpha_I$ where $\alpha_I = \alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k}$.

Since $\alpha_I(\beta_J) = \begin{cases} 0 & I \neq J \\ 1 & I = J \end{cases}$ we see $\eta = \omega$

and α_I are linearly independent. $\blacksquare$

<u>Note</u>: ① $\alpha_i \in V^*$ then $\alpha_{\sigma(1)} \wedge \cdots \wedge \alpha_{\sigma(k)}$

$$= (-1)^{\operatorname{sgn}(\sigma)} \alpha_1 \wedge \cdots \wedge \alpha_k$$

② Have $\Lambda^k(V) \times \Lambda^\ell(V) \xrightarrow[\wedge]{} \Lambda^{k+\ell}(V)$

which in particular ~~set~~ set

$$(\alpha_1 \wedge \cdots \wedge \alpha_k) \wedge (\beta_1 \wedge \cdots \wedge \beta_\ell)$$
$$= \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \beta_\ell$$

Props of wedge (exterior) product: (14.11 of Lee)

$\omega, \omega' \in \Lambda^k(V) \qquad \eta, \eta' \in \Lambda^l(V) \qquad a, a' \in \mathbb{R}$

Bilinear: $(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta)$

$\omega \wedge (a\eta + a'\eta') = a(\omega \wedge \eta) + a'(\omega \wedge \eta')$

Assoc: $\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$

Anticomm: $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$

Note: Can/Should define the wedge product more intrinsically:

$$\omega \wedge \eta (v_1, v_2, \ldots, v_{k+l})$$

$$= \underbrace{\frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\, \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})}_{\in \Lambda^{k+l}(V)}$$

Compare: $\det(a_{ij}) = \sum_{\sigma \in S_n} (\operatorname{sgn} \sigma)\, a_{1\sigma(1)} \cdots a_{n\sigma(n)}$.