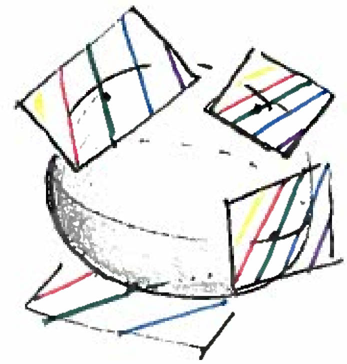
Lecture 22: Covector fields

V vector space $\quad V^* = \{f : V \to \mathbb{R} \mid f \text{ linear transformation}\}$

Cotangent bundle: $T^*M = \coprod_p T_p^*M$ where $T_p^*M = (T_pM)^*$

Covector field: $(p \in M) \longmapsto (\omega_p \in T_p^*M)$.



$$\omega = xy\, dx + e^{x^2+y}\, dy$$

[Just as for vector fields, there is a notion of smoothness for covector fields...]

$$\Omega^1(M) = \left\{ \begin{array}{l} \text{smooth covector} \\ \text{fields on } M \end{array} \right\}$$

- Vector space
- $C^\infty(M)$-module

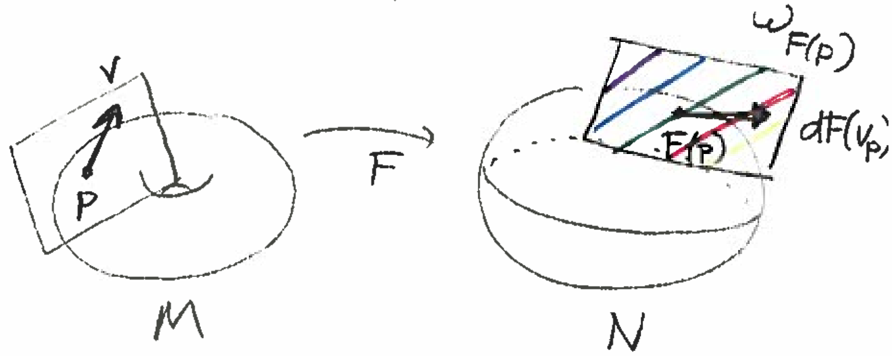
Also called "differential 1-forms".

Pull backs: $F : M \to N$ smooth. For $\omega \in \Omega^1(N)$

define $F^*(\omega) \in \Omega^1(M)$ by $(F^*\omega)_p = (dF_p)^*(\omega_{F(p)})$

that is $(F^*\omega)_p(v \in T_pM)$

$= \omega_{F(p)}(dF_p(v))$



Unlike vector fields, where $F_* X$ only sometimes makes sense, can <u>always</u> pull back a covector field.

[Query: Is this a diffeo?]

<u>Ex</u>: $F : \underset{(x,y)}{\mathbb{R}^2} \longrightarrow \underset{(u,v)}{\mathbb{R}^2}$ $F(x,y) = (x+y^2, x^2)$

$\omega = v\,du + dv$ $DF = \begin{pmatrix} 1 & 2y \\ 2x & 0 \end{pmatrix}$

$$(F^*\omega)_{(x,y)}\left(\frac{\partial}{\partial x}\right) = \omega_{(x+y^2,\, x^2)}\left(dF_{(x,y)}\left(\frac{\partial}{\partial x}\right)\right)$$

$$= \omega_{(\underset{u}{x+y^2},\, \underset{v}{x^2})}\left(\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}\right)$$

$$= x^2 + 2x$$

$$(F^*\omega)_{(x,y)}\left(\frac{\partial}{\partial y}\right) = \omega_{(x+y^2,\, x^2)}\left(2y\frac{\partial}{\partial u}\right)$$

$$= 2yx^2$$

So $F^*\omega = (x^2+2x)\,dx + 2yx^2\,dy$

<u>Check</u> (Uses HW!) $(F^*\omega)_{(x,y)} = \left(dF_{(x,y)}\right)^*\left(\omega_{(x+y^2,\, x^2)}\right)$

$$= \underbrace{\begin{pmatrix} 1 & 2x \\ 2y & 0 \end{pmatrix}}_{\text{transpose}}\begin{pmatrix} x^2 \\ 1 \end{pmatrix} = \begin{pmatrix} x^2+2x \\ 2yx^2 \end{pmatrix} \checkmark$$

For $M \xrightarrow{f} N \xrightarrow{g} S$ have

$$\begin{tikzcd} \Omega^1(M) & \Omega^1(N) \arrow[l, "f^*"'] & \Omega^1(S) \arrow[l, "g^*"'] \arrow[ll, bend left, "f^* \circ g^* = (g\circ f)^*"] \end{tikzcd}$$

———— ∘ ————

For $f : M \to \mathbb{R}$, the <u>differential</u> $\omega_f \in \Omega^1(M)$ is defined by $\omega_f(V_p \in T_pM) = V_p(f)$.

[This is the "gradient-like" covector field from Wed.]

For $G : N \to M$, we have $\boxed{\omega_{f\circ G} = G^*(\omega_f)}$ since

$$\omega_{f\circ G}(v_q) = V_q(f\circ G) = \big(dG_q(V_q)\big)f$$

$$= (\omega_f)_{G(q)}\big(dG_q(v_q)\big) = (G^*\omega_f)(v_q)$$

<u>Thm</u>: $f : M \to \mathbb{R}$ smooth. Then $\omega_f = f^*(dt)$ where $dt \in \Omega^1(\mathbb{R})$ is the dual to $\frac{\partial}{\partial t}$.

<u>Pf</u>: Let $i : \mathbb{R} \to \mathbb{R}$ be the identity map. Theorem holds for $i : t \mapsto t$ since

$$\omega_i\left(a \frac{\partial}{\partial t}\Big|_p\right) = \left(a\frac{\partial}{\partial t}\Big|_p\right)(i) = a = dt\left(a\frac{\partial}{\partial t}\Big|_p\right)$$

and $i^*(dt) = dt$. The result for general $f : M \to \mathbb{R}$ now follows by ✪ since

$$\omega_f = \omega_{i \circ f} = f^*(\omega_i) = f^*(dt). \qquad \blacksquare$$

<u>Notation:</u> The usual notation for ω_f is df.

Unfortunately, this is <u>also</u> our notation for the derivative $TM \xrightarrow[df]{} T\mathbb{R}$. Both ω_f and our original df "eat" tangent vectors, but the former outputs a elt of $\mathbb{R}$ and the latter an elt of $T_{f(p)}\mathbb{R}$

Specifically,

$$df(v_p) = \omega_f(v_p) \frac{\partial}{\partial t}\Big|_{f(p)}$$

or

$$\begin{tikzcd} TM \arrow[r, "df"'] \arrow[rr, bend left, "\omega_f"] & T\mathbb{R} \arrow[r, "\pi"] & \mathbb{R} \end{tikzcd}$$

$$(p, a\tfrac{\partial}{\partial t}|_p) \mapsto a$$

From now on, will denote ω_f by df (since this the standard notation.) and yet continue to denote $TM \to T\mathbb{R}$ by df.

[Blame Lee for this mess; other books use DF or F_* for the derivative....]

<u>Note</u>: On $\mathbb{R}^n$ with coor $(x_1, \ldots, x_n)$ then

ω_{x_i} = "formal dx_i"

← dual basis to $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$

<u>Integration</u>: $[a,b] \subseteq \mathbb{R}$ bounded interval [A manifold with boundary

For $\omega \in \Omega^1([a,b])$ define $\int_{[a,b]} \omega = \int_a^b f(t)\,dt$

where $\omega = f(t)\,dt$

<u>Thm</u>: Suppose $F : [c,d] \longrightarrow [a,b]$ is a diffeomorphism. Then $\forall \omega \in \Omega^1([a,b])$ one has

$$\int_{[a,b]} \omega = \int_{[c,d]} F^*(\omega)$$