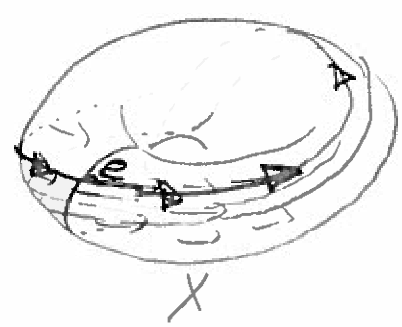
Lecture 20: 1-parameter subgroups and the expontial map.

Suppose H is a Lie subgp of G. Then $\mathfrak{h}$ is a sub-Lie algebra of $\mathfrak{g}$ via $T_eH \subseteq T_eG$.

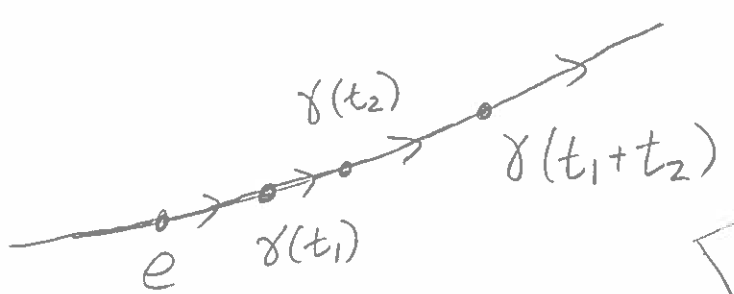
Fact: Any sub Lie algebra of $\mathfrak{g}$ comes from some Lie subgp. [Today: Last lecture on Lie gps for awhile.]

G a Lie gp. If $L \subseteq \mathfrak{g}$ is a 1-dim'l vector subsp, then L is a sub Lie algebra, [Query: Why?] since $[sX, tX] = st[X,X] = 0$. [So expect a 1-dim'l subgp of G corresponding to L.] Let X in $\mathfrak{g}$ be nonzero. Let $\gamma : \mathbb{R} \longrightarrow G$ be the integral curve of X with $\gamma(0) = e$.



Claim: γ is a Lie gp homomorphism.

Pf: Let $t_1, t_2 \in \mathbb{R}$. Must show $\gamma(t_1+t_2) = \gamma(t_1)\cdot\gamma(t_2)$



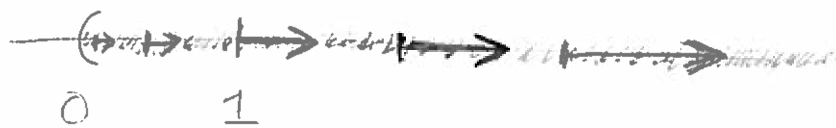
$\boxed{\gamma \text{ is a "1-parameter subgp of } G\text{"}}$

Now $\gamma(t_1)\cdot\gamma(t_2) = L_{\gamma(t_1)}(\gamma(t_2))$,

and $L_{\gamma(t_1)} \circ \gamma$ is the integral curve for $(L_{\gamma(t_1)})_* X = X$ starting at $\gamma(t_1)$. Hence $(L_{\gamma(t_1)} \circ \gamma)(t) =$

$\gamma(t_1 + t)$ for all t; in particular, $\gamma(t_1+t_2) = \gamma(t_1)\cdot\gamma(t_2)$

Ex: $G = (\mathbb{R}_+, \times)$



Left inv. vector field assoc to $\left.\frac{\partial}{\partial x}\right|_1$ is $X_a = a\left.\frac{\partial}{\partial x}\right|_a$ since $L_a : x \longmapsto ax$. Integral curve is

$t \longmapsto e^t$ a gp homomorphism.

Ex: $G = GL_2\mathbb{R}$ $\qquad \mathfrak{g} = T_{Id}G = M_2(\mathbb{R}) \cong \mathbb{R}^4$

$A \in G$

$$\begin{array}{ccccccc} dL_A : & T_{Id}G & \longrightarrow & T_A G & & & \\ & \shortparallel & & \shortparallel & & & \\ & M_2(\mathbb{R}) & & M_2(\mathbb{R}) & G & & G \\ & & & & \cap\! \mid & & \cap\! \mid \end{array}$$

is just $X \longmapsto AX$. Reason: $L_A : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is a linear transformation. Concretely

$$\underset{A}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} u & v \\ w & x \end{pmatrix} = \begin{pmatrix} au+bw & av+bx \\ cu+dw & cv+dx \end{pmatrix}$$

some elt of G

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}\begin{pmatrix} u \\ v \\ w \\ x \end{pmatrix}$$

① $X_{Id} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $X_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$. Integral curve is

$$\gamma(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in GL_2\mathbb{R} \qquad \gamma'(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}} \quad \forall t$$

② $X_{Id} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $X_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} a & -b \\ c & -d \end{pmatrix}$ $\gamma(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

$$\gamma'(t) = \begin{pmatrix} e^t & 0 \\ 0 & -e^{-t} \end{pmatrix}$$

③ $X_{Id} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $X_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}$

$$\gamma(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \qquad \gamma'(t) = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix}$$

[These seem quite disconnected, but they're not!]

Define $M_n(\mathbb{R}) \xrightarrow[\exp]{} GL_n\mathbb{R}$ by

$$e^X = 1 + X + \frac{1}{2}X^2 + \frac{1}{6}X^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

Thm: The above converges to a matrix in $GL_n\mathbb{R}$ for any $X \in M_n(\mathbb{R})$. Moreover, $X \mapsto e^X$ is smooth and $\gamma(t) = e^{tX}$ is the 1-param. subgp of $GL_n\mathbb{R}$ with $\gamma'(0) = X$.

[Check examples (1-3) above using this.]

In general, for G one defines

$$\exp : \mathfrak{g} \longrightarrow G$$

by $\exp(X) = \gamma(1)$ where γ is the integral curve of X with $\gamma(0) = e_G$. This turns out to be smooth and is e^X for $G = GL_n\mathbb{R}$.

<u>Important</u>: Typically, exp is <u>not</u> a gp homomorphism. It does sat $\exp((s+t)X) = \exp(sX)\cdot\exp(tX)$, though.

<u>Thm</u>: G a Lie gp. If $\mathfrak{h}$ is a Lie sub alg. of $\mathfrak{g}$, there exists a unique connected Lie subgp H with $T_eH = \mathfrak{h}$. Moreover, H is normal iff $\mathfrak{h}$ is an ideal, i.e. $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.