Lecture 16: Vector fields, integral curves, and flows.

$\Theta : \mathbb{R} \times M \to M$ smooth action

$(t, m) \longmapsto t \cdot m = \Theta_t(m)$

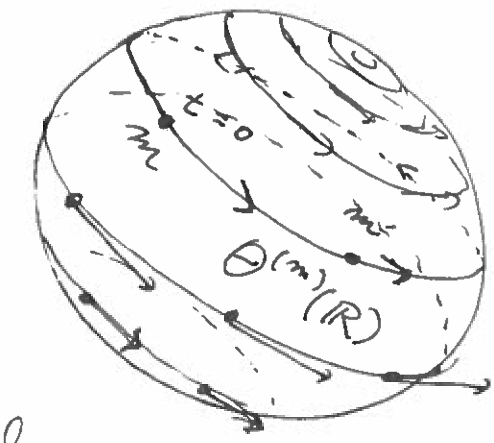
$\Theta_t : M \to M$ diffeomorphisms with $\Theta_{t_1} \circ \Theta_{t_2} = \Theta_{t_1 + t_2}$

For $m \in M$ have a curve $\Theta^{(m)} : \mathbb{R} \to M$

$t \longmapsto \Theta_t(m)$

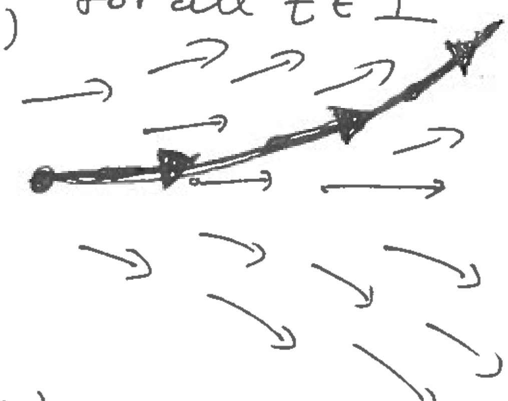
<u>Infinitessimal generator</u>: $V \in \mathfrak{X}(M)$

$$V_m = \frac{d}{dt} \Theta^{(m)} \Big|_{t=0} = d\Theta\left(\frac{\partial}{\partial t}\Big|_{(0,m)} \right)$$



A curve $\gamma : I \to M$ is an <u>integral curve</u> for $X \in \mathfrak{X}(M)$ if $\gamma'(t) = X_{\gamma(t)}$ for all $t \in I$

[γ a solution to an ODE on M.]



<u>Ex</u>: Smooth action $\Theta : \mathbb{R} \times M \to M$,

V the inf. gen. Then each $\Theta^{(m)}$ is an integral curve for V as follows:

Set $m' = \theta^{(m)}(t)$; since $\theta^{(m')}(s) = \theta^{(m)}(s+t)$ ②

$$\theta^{(m)\prime}(t) = \left(\theta^{(m')}\right)'(0) = V_{m'} = V_{\theta^{(m)}(t)}$$

Note: $\mathbb{R}$-actions are also called *flows*.

Does every $X \in \mathfrak{X}(M)$ come from a flow?

No: $M = \{ x \in \mathbb{R}^2 \mid |x| < 1 \}$

$$X = \frac{\partial}{\partial x}$$

Integral curve containing $(0,0)$:

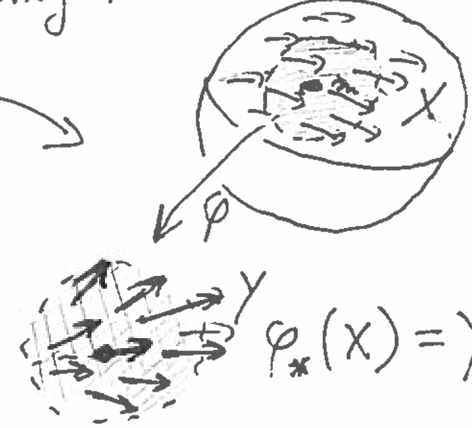
$$\begin{aligned} \gamma : (-1,1) &\longrightarrow M \\ t &\longmapsto (t,0) \end{aligned} \qquad \gamma'(t) = \frac{\partial}{\partial x} \quad \checkmark$$

can't enlarge the domain so can't set $\theta^{((0,0))} = \alpha$.

Thm: $X \in \mathfrak{X}(M)$. For each $m \in M$ there is open interval $I(m)$ containing 0 and a curve $\gamma : I(m) \to M$ where ⓐ γ is an integral curve for X with $\gamma(0) = m$.

ⓑ If $\alpha : J \to M$ is an int curve for X with $\alpha(0) = m$ then $J \subseteq I(m)$ and $\alpha = \gamma\big|_{I(m)}$.

<u>Pf</u>: Given $m \in M$, the existence of an integral curve on some $(-\varepsilon, \varepsilon)$ follows from applying the existence thm for ODE's in some chart.



$\varphi_*(X) = Y$

$$\underbrace{\gamma'(t) = Y_{\gamma(t)}}_{}$$

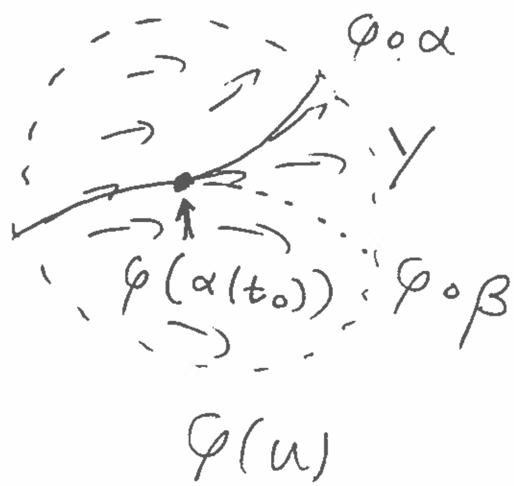
system of n first-order ODE's.

Suppose $\alpha : I \to M$ and $\beta : I \to M$ are integral curves with $\alpha(0) = \beta(0) = m$.

<u>Claim</u>: $\alpha = \beta$ on $I \cap J$.

[If true then we just define γ by taking $I(m) = \bigcup \{\text{domain of some } \alpha\}$]

Let $t_0 \in I \cap J$ be such that $\alpha(t_0) = \beta(t_0)$ but $\exists\, t$ arb. close to t_0 with $\alpha(t) \neq \beta(t)$. In local coor near $\alpha(t_0)$ have



Contradicts <u>uniqueness</u> of solutions to ODE's. ▨

Let $X \in \mathfrak{X}(M)$. Define

$$\mathcal{D} = \{(t,m) \in \mathbb{R} \times M \mid t \in I(m)\}$$

and $\Theta : \mathcal{D} \to M$ by $(t,m) \longmapsto \gamma_m(t)$

where $\gamma_m : I(m) \longrightarrow M$ is the int. curve for X

where $\gamma_m(0) = m$.

<u>Thm</u>: $\mathcal{D}$ is open in $\mathbb{R} \times M$ and Θ is smooth

<u>Reason</u>: Smooth dep. of solutions to ODE's on initial conditions.

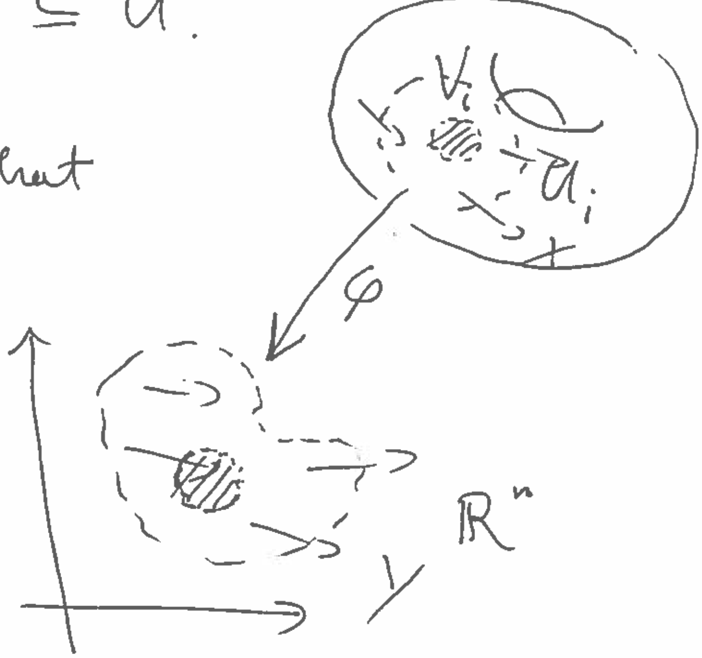
<u>Complete vector field</u>: $X \in \mathfrak{X}(M)$ where $\mathcal{D} = \mathbb{R} \times M$.

[Precisely those coming from $\mathbb{R}$-actions]

<u>Thm</u>. If M is compact, then any $X \in \mathfrak{X}(M)$ is complete.

<u>Pf</u>. Cover M with finitely many V_i where each $V_i \subseteq (U_i, \varphi_i)$ and the closure of

$\varphi_i(V_i)$ in $\mathbb{R}^n$ is cpt an $\subseteq \mathcal{U}$.

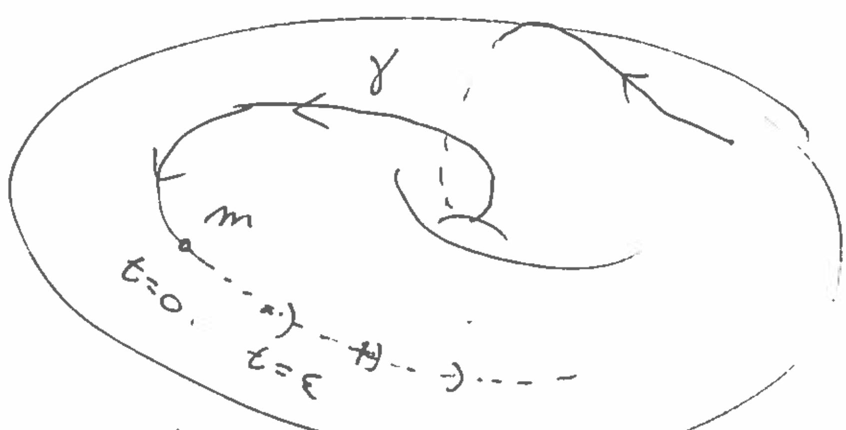


By ODE theory, $\exists\, \varepsilon_i$ so that

$\gamma' = Y_\gamma$ has sol on $(-\varepsilon_i, \varepsilon_i)$

for all init. cond in $\overline{\varphi(V_i)}$.

So back on M, any integral curve can be extended by at least time $\varepsilon = \min \varepsilon_i > 0$. So M is complete. $\blacksquare$



<u>Thm</u>: Every left-invariant vector field on a lie gp G is complete.

<u>Pf</u>: Integral curve exists at e for some time $\pm\varepsilon$. By left invariance this is true at every other $g \in G$. So the left-inv. v.f. is complete.