Lecture 29: Partitions of Unity

Goal: Define $\int_M \omega$ where $\omega \in \Omega^n(M)$ and M is an n-manifold.

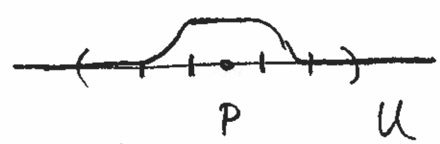
Def: The support of $f: M \to \mathbb{R}$ is $\overline{\operatorname{supp} f = \{p \in M \mid f(p) \neq 0\}}$

HW #2: Given $p \in M$ there is a smooth chart (U, φ) about p and a smooth $f: M \to [0,1]$ where

(a) $f = 1$ on a nbhd of p

(b) $\overline{\operatorname{supp} f} \subseteq U$.

Bump function for (U, φ)

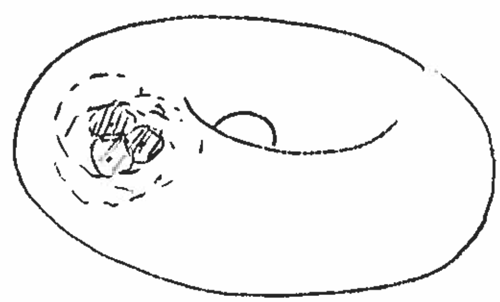


Thm: Every smooth M^n has a countable set of smooth charts (U_i, φ_i) with bump functions f_i so that:

(a) $\bigcup \operatorname{supp} f_i = M$

(b) Any $p \in M$ is in finitely many $\operatorname{supp} f_i$.

Cor. Any smooth M^n has a Riemannian metric.



<u>Pf of Cor:</u> Let (U_i, φ_i, f_i) be as in the thm. Define $(g_i)_p : T_pM \times T_pM \to \mathbb{R}$ by

$$g_i(v_p, w_p) = \begin{cases} 0 & \text{if } p \notin \operatorname{supp} f_i \\ f(p)\, g_{DOT}(d\varphi_i(v_p), d\varphi_i(w_p)) & \text{if } p \in \operatorname{supp} f_i \end{cases}$$

[Query: Is g_i a Riemannian metric?]

Not quite a Riemannian metric,

satisfies everything except pos. def;

Do have $g_i(v_p, v_p) \geq 0$ though.

Define $g : T_pM \times T_pM$ by

$$g(v_p, w_p) = \sum_i g_i(v_p, w_p)$$

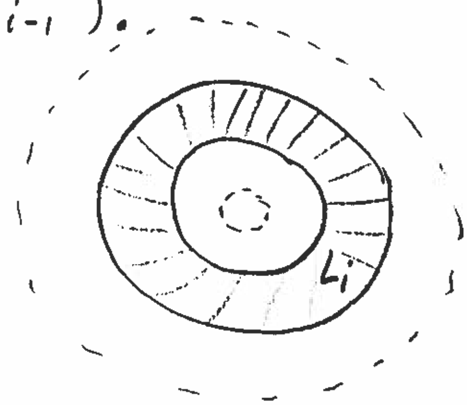
which makes sense because of (b). Its an actual Riemannian metric since some term in $g(v_p, v_p)$ is > 0. ∎

Lemma: A smooth M^n is a countable union of compact K_i where $K_i \subseteq \mathrm{Int}(K_{i+1})$

[Query: What if $M = \mathbb{R}^n$?]

Pf: M is second countable and locally compact. See A.60 in Lee. ▨

Pf of Thm: Let $L_i = K_i \setminus \mathrm{Int}(K_{i-1})$.



By compactness, $\exists$ finitely many $(U_\alpha, \varphi_\alpha, f_\alpha)$ so that

(i) Each $U_\alpha \subseteq \mathrm{Int}(K_{i+1}) \setminus K_{i-1}$

(ii) The supp f_α cover L_i

The union of these as i varies is what we seek; condition (b) holds since the U_α for L_i meet at most L_{i+1} and L_{i-1}.

<u>Thm</u>: Every smooth M^n has a countable set of smooth charts (U_i, φ_i) and $\psi_i \in C^\infty(M)$ where

Partition of Unity

1. Any $p \in M$ is in finitely many $\operatorname{supp} \psi_i$
2. $\overline{\operatorname{supp} \psi_i} \subseteq U_i$
3. For all $p \in M$
$$\sum_i \psi_i(p) = 1.$$

<u>Pf</u>: Let (U_i, φ_i, f_i) be as in the first thm.

Define $f \in C^\infty(M)$ by $f(p) = \sum_i f_i(p)$.

[Makes sense by (b)]. Since $f(p) > 0$ [by (a)] set

$\psi_i = f_i/f$. Then $\sum \psi_i = 1$. ▨

<u>Motivation</u>: Break $\int_M \omega$ into $\int_M (\sum_i \psi_i)\omega = \sum_i \int_M \psi_i \omega$ and evaluate the latter as $\int_{\varphi_i(U_i)} (\varphi^{-1})^*(\psi_i \omega)$.