~ecture   Calculus to Cohomology

Original goal: Do calculus on things locally like $\mathbb{R}^n$.

[Derivatives, vector fields, differential forms, Lie bracket, Lie derivatives...]

So far our understanding of manifolds has been based on their local properties. Indeed, the ~~[illegible]~~ most central concepts of this course has been how to define globally objects we understand on $\mathbb{R}^n$.

New focus: How do we use these tools to understand/distinguish the global topology of mflds?

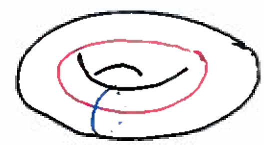
None of these are diffeomorphic

Showing
Existence can be easier than non-existence.

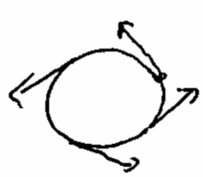
Ex: $S^1 \times S^1 \subseteq \mathbb{R}^4$ is diffeom to $\{(\sqrt{x^2+y^2}-2)^2+z^2=1\}$

$\quad \parallel$

$\{(u,v,w,x) \mid u^2+v^2=w^2+x^2=1\}$



Ex: S^1 and S^3 have nowhere-vanishing vector fields



S^1

$\mathbb{C}^2 \supseteq S^3$ has infinitesimal gen of mult by e^{it}

Thm:  and  are not diffeomorphic.

Thm: Every $X \in \mathfrak{X}(S^2)$ vanishes at at least one pt.

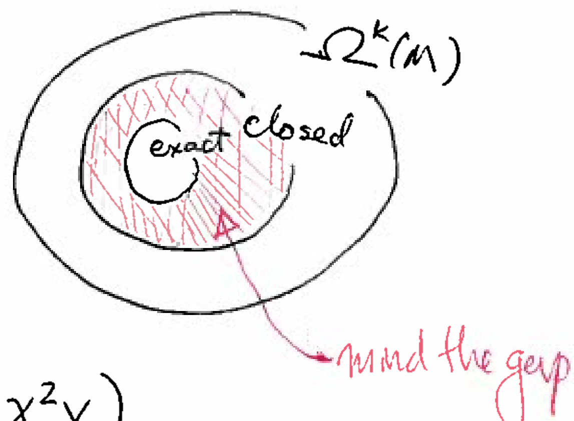
Thm: $D^n = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$. There does not exist a smooth map $f: D^n \to \partial D^n$ where $f|_{\partial D^n} = id|_{\partial D^n}$

[Of course the distinction between existence and non-existence is fuzzy.]

Thm: Suppose $F: D^n \to D^n$ is smooth. Then $\exists p \in D$ with $F(p) = p$.

A form $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$; means zero at every point of M

it is exact if $\exists \eta \in \Omega^{k-1}$ with $d\eta = \omega$.



Note: Since $d \circ d = 0$ have:

① On $\mathbb{R}^2$:

$$\omega = 2xy\, dx + x^2 dy$$

is closed and exact since $\omega = d(x^2 y)$

the form

$$\omega = y\, dx \text{ is neither closed or exact.}$$

② Suppose $M \underset{\text{open}}{\subseteq} \mathbb{R}^3$. $\quad \Omega^1(M) \longleftrightarrow \mathfrak{X}(M)$

$$\omega = a\,dx + b\,dy + c\,dz \longleftrightarrow a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} = V$$

$$\omega \text{ closed} \iff \operatorname{curl} V = 0$$

$$\omega \text{ exact} \iff V = \operatorname{grad} f \quad \text{for } f \in C^\infty(M)$$

(V conservative)

Forshadowing: When $M = \mathbb{R}^3$ then closed and exact are equivalent.

③ Suppose M has no boundary and is orientable, Then any $\omega \in \Omega^n(M)$ is closed, and if ω is exact then $\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0$

So if $\omega \in \Omega^2(S^2)$ is the volume form on S^2 [from HW.], then $\int_M \omega = 4\pi$ and so ω is not exact.

Suppose $\omega \in \Omega^k(M)$ is closed but not exact.

Then so is $\omega + d\eta$ for every $\eta \in \Omega^{k-1}(M)$.

Note: $\begin{Bmatrix}\text{exact}\\ \text{forms}\end{Bmatrix} \subseteq \begin{Bmatrix}\text{closed}\\ \text{forms}\end{Bmatrix} \subseteq \Omega^k(M)$ are linear subspaces, so can define the k^{th}-deRahm cohomology group as

$$H^k(M) = \{\text{closed}\}/\{\text{exact}\}.$$

That is, $H^k(M)$ is the set of equiv. classes $[\omega]$ where $d\omega = 0$ and $[\omega] = [\omega']$ if $\omega - \omega'$ is exact.

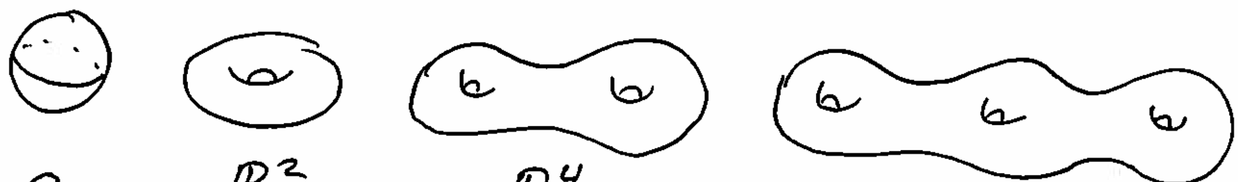
Fact: When M is cpt, this actually a finite-dim'l vector space.

Note if you know about simplicial/singular/cellular cohomology, this is $H^k(M;\mathbb{R})$.

Ex: $H^k(\mathbb{R}^n) = \begin{cases} 0 & k>0 \\ \mathbb{R} & k=0 \end{cases}$ call back to vector fields on $\mathbb{R}^3$.

Ex: Suppose M is a compact connected orientable n-mfld. Then $H^n(M) \cong H^0(M) \cong \mathbb{R}$.

Ex:



H^1_g 0 $\mathbb{R}^2$ $\mathbb{R}^4$ $\mathbb{R}^6$.

Basic properties:

① Set $H^*(M) = \bigoplus_{k=0}^{n} H^k(M)$. Then $H^*(M)$ is an algebra via the multiplication $[\alpha]\wedge[\beta] = [\alpha\wedge\beta]$

Makes sense because $(\alpha + d\eta)\wedge\beta = \alpha\wedge\beta + d\eta\wedge\beta$

$$= \alpha\wedge\beta + d(\eta\wedge\beta)$$

since $d\beta = 0$; same for other poss. reps of $[\beta]$.

② If $F: M \to N$ get a map $F^*: H^*(N) \longrightarrow H^*(M)$ from the associated map $\Omega^*(N) \longrightarrow \Omega^*(M)$ since d and F commute.

(3) Cor: If M and N are diffeomorphic, then $H^*(M) \cong H^*(N)$ (as $\mathbb{R}$-algebras).

(4)

Prop: Suppose $M_1, \ldots, M_n$ are smooth mfds. Then $H^*\left(\coprod_{i=1}^n M_i\right) \cong \left(\prod \text{ or } \bigoplus_{i=1}^n\right) H^*(M_i)$

(5) Prop: If M^n is connected then $H^0(M) = \mathbb{R}$.

Pf: There are no exact forms since $\Omega^{-1} = 0$. ~~then~~ A function $f \in \Omega^0(M)$ ~~is~~ has $df = 0$ exactly when it is constant. The subspace of constant functions is $\cong \mathbb{R}$. ∎