Lecture 32:

Thm: M smooth. There are unique maps $d: \Omega^k(M) \to \Omega^{k+1}(M)$ satisfying: (a) d is linear $/\mathbb{R}$

(b) $\omega \in \Omega^k(M)$ and $\eta \in \Omega^\ell(M)$:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$

(c) $d \circ d$

(d) For $f \in \Omega^0(M) = C^\infty(M)$, the $df \in \Omega^1(M)$ is the usual differential, i.e. $df(V_p) = V_p f$.

One reason for $(-1)^k$ in (b): Needed so that

$$d(\eta \wedge \omega) = (-1)^{k\ell} d(\omega \wedge \eta) \text{ to match } \eta \wedge \omega = (-1)^{k\ell} \omega \wedge \eta.$$

Lie derivatives: $V \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$.

Define $\mathcal{L}_V \omega \in \Omega^k(M)$ by

$$(\mathcal{L}_V \omega)_p = \frac{d}{dt} \underbrace{(\theta_t^* \omega)_p}_{\text{in } \Lambda^k T_p M} \Big|_{t=0}$$

where θ is the flow associated to V.

Using the proof that $L_X Y = [X, Y]$

this is the same as defining $\mathcal{L}_V \omega$ by the property that for any $X_1, \ldots, X_k \in \mathcal{X}(M)$ one has

$$\mathcal{L}_V \omega(X_1, \ldots, X_k) = V(\omega(X_1, \ldots, X_k))$$
$$- \omega([V, X_1], X_2, \ldots, X_k) - \cdots$$
$$- \omega(X_1, X_2, \ldots, [V, X_k])$$

<u>Prop</u>: $V \in \mathcal{X}(M)$ and $\omega, \eta \in \Omega^*(M)$. Then

$$\mathcal{L}_V(\omega \wedge \eta) = (\mathcal{L}_V \omega) \wedge \eta + \omega \wedge (\mathcal{L}_V \eta)$$

[No signs needed here as $\mathcal{L}_V$ doesn't change the degree of the forms. $\mathcal{L}_V$ and d are related:]

<u>Cartan's Magic Formula</u>:

$$\mathcal{L}_V \omega = V \lrcorner\, d\omega + d(V \lrcorner\, \omega)$$

Here, $V \lrcorner\, \eta$ is the <u>interior product</u> defined by

$$(V \lrcorner\, \eta)(\underbrace{W_1, \ldots, W_{k-1}}_{\in T_pM}) = \eta(V, W_1, \ldots, W_{k-1})$$

<u>Stokes Thm</u>: Let M be an oriented smooth n-manifold with boundary. If $\omega \in \Omega^{n-1}(M)$ is compactly supported, then $\int_M d\omega = \int_{\partial M} \omega$.

<u>Recall</u>: Such an M has charts to open sets in $\mathbb{R}^n$ and $H^n = \{x \in \mathbb{R}^n \mid x_n \geq 0\}$. $\{x \in H^n \mid x_n = 0\}$

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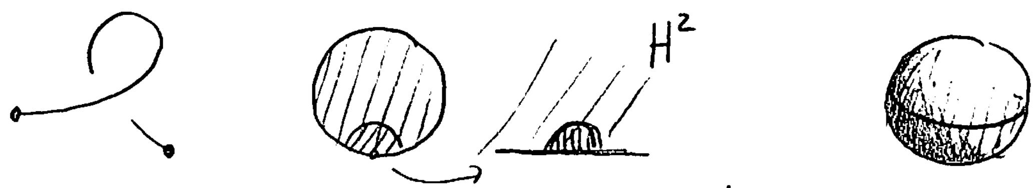
Then $\partial M = \{p \in M \mid \exists \text{ smooth chart } (U, \varphi) \text{ with } \varphi(p) \in \partial H^n\}$



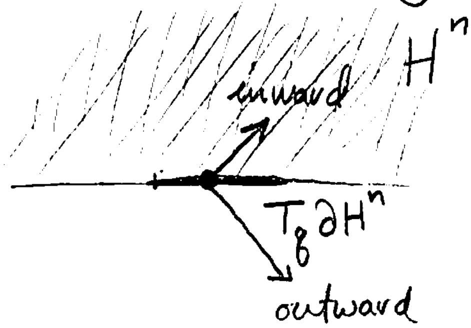
$= \{x \in \mathbb{R}^3 \mid |x| \leq 1\}$

Note ∂M is itself a smooth manifold (with the topology inherited from M) <u>without</u> boundary.

<u>Prop</u>: An orientation of M induces one of ∂M. [In particular, ∂M is orient when M is.]

For $p \in \partial M$, the tangent space T_pM is still $\mathbb{R}^n$; you can view it as ident with $T_{\varphi(p)}H^n = T_{\varphi(p)}\mathbb{R}^n \cong \mathbb{R}^n$. [Alt, its derivations of smooth fns at p.]

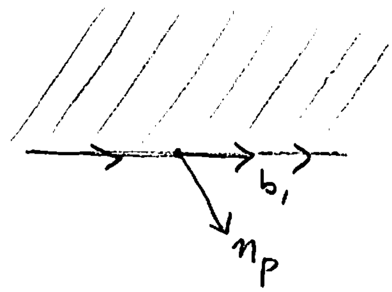
Given $v \in T_p M$ for $p \in \partial M$ have one of (a) $v \in T_p \partial M$ (b) v inward pointing (c) outward pointing



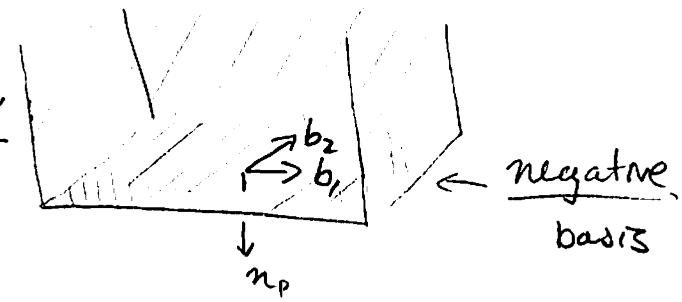
Given an orientation of M, orient ∂M by the following rule.

A basis $b_1, \ldots, b_{n-1} \in T_p \partial M$ is positively oriented if when $n_p \in T_p M$ is outward pointing, then $n_p, b_1, \ldots, b_{n-1}$ is a pos. basis for $T_p M$.

Ex:



Ex



So ∂H^n gets the standard orient of $\mathbb{R}^{n-1}$ when n is even and the opposite orient when n is odd

[To prove the prop, have to check that the pointwise orient defined above is locally consistent. But thats clear from the picture for H^n.]

[Go back to statement of Stokes thm.]

<u>Special cases</u>: (1) If $\partial M = \emptyset$, view $\int_{\partial M} \omega$ as 0.

(2) If $\dim M = 1$, then ∂M = some points.

An orient of a point mfld p is just a sign $\varepsilon_p = \pm 1$ and $\int_p f = \varepsilon_p f(p)$.

(3) So if $M = [a,b]$ and $f \in \Omega^0(M)$ then

$$\int_M df = \int_a^b f'(t)\,dt = f(b) - f(a) = \int_{\partial M} f$$

[If time remains, blather about how this connects to Math 241.]