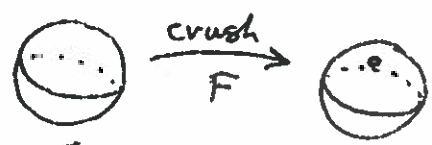
Lecture 40: Degrees of maps of spheres.

Last time: $H^k(S^n) = \begin{cases} \mathbb{R} & k = 0, n \\ 0 & \text{otherwise} \end{cases}$

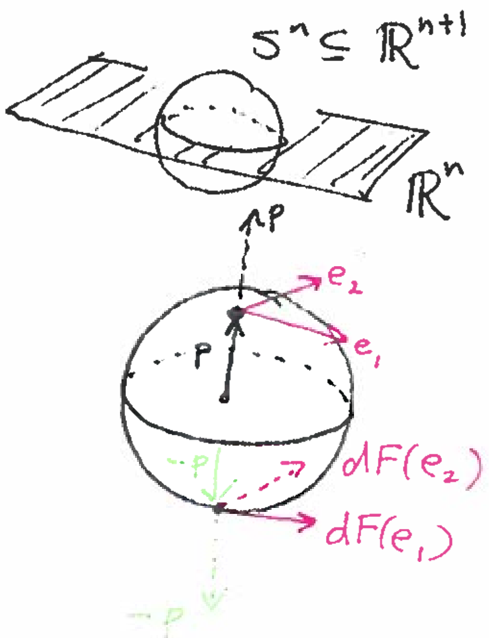
The degree of a smooth $F: S^n \to S^n$ is the $\deg(f) \in \mathbb{R}$ so that $F^*([\omega]) = \deg(f)[\omega]$ for all $[\omega] \in H^n(S^n) \cong \mathbb{R}$.

Ex: (a) $\deg(id_{S^n}) = 1$ (b) $\deg(\text{const map}) = 0$



$F^*\omega = 0$ $\omega \in \Omega^n(S^n)$

(c) $\deg(\text{Reflect in } \mathbb{R}^n) = -1$



$F(x_1, \dots, x_{n+1}) = (x_1, \dots, x_n, -x_{n+1})$

$F^*(\text{Volume form } \omega_g) = -\omega_g$

(d) $S^1 \xrightarrow{F} S^1$ $d\theta$ gen of $H^1(S^1)$

$z \longmapsto z^2$

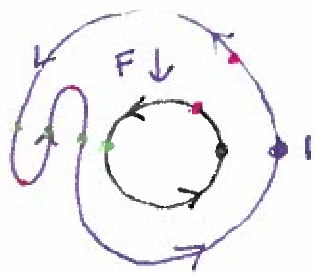
$F^*(d\theta) = 2\,d\theta \Rightarrow \deg F = 2$

(e) $S^1 \longrightarrow S^1$ has $F^*(d\theta) = n\,d\theta \Rightarrow \deg F = n$.

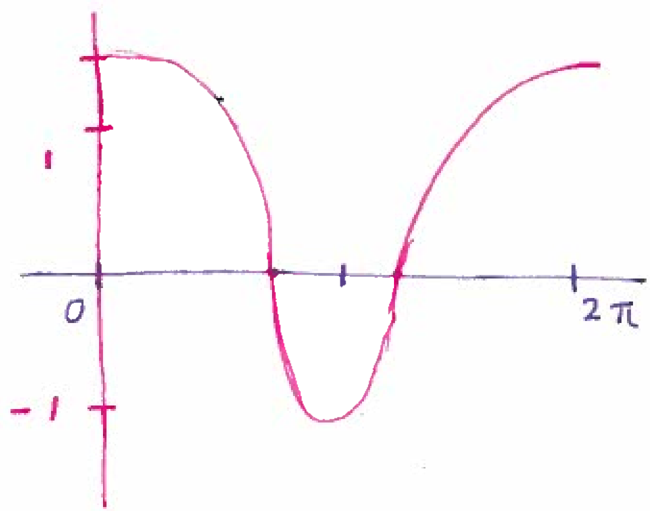
$z \longmapsto z^n$

<u>Note</u>: Lucking out that $F^*(\omega) = (\deg f)\,\omega$; typically this only happens on the level of cohomology.

$F: S^1 \to S^1$



$F^*(d\theta) = g(\theta)\, d\theta$



<u>Note</u> $H^n(S^n) \xrightarrow{\cong} \mathbb{R}$

$[\omega] \longmapsto \int_{S^n} \omega$

So if $[\omega] \neq 0$ have

$$\deg F = \frac{\int_{S^n} F^*\omega}{\int_{S^n} \omega}$$

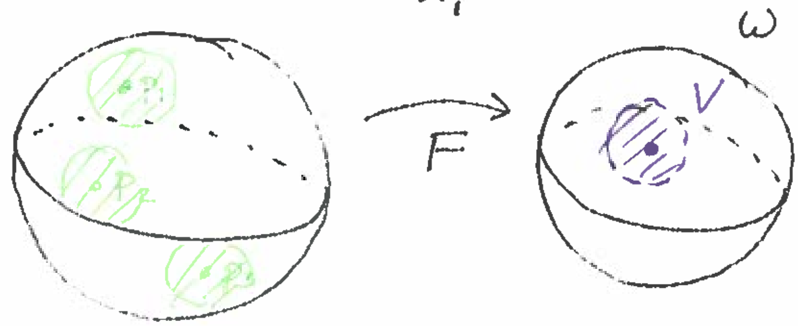
In this case F is homotopic to id $\Rightarrow$ deg $= 1$.

<u>Thm</u>: For any smooth $F: S^n \to S^n$, $\deg F \in \mathbb{Z}$. For any regular value $q \in S^n$ we have

$$\deg F = \sum_{p \in F^{-1}(q)} \begin{cases} +1 & \text{if } dF_p \text{ is orient pres.} \\ -1 & \text{if } dF_p \text{ is orient rev.} \end{cases}$$

Note: Homotopic maps have the same degree. In fact, the converse is true as well.

Proof: Let $q \in S^n$ be a regular value of F. Then $F^{-1}(q)$ is an embedded submfld of dim 0, i.e. a finite set of points $p_1, \ldots, p_k$. Can choose disjoint open nbhds U_i of p_i and V of q so that each $F|_{U_i}$ is a diffeo onto V.



Pick $\omega \in \Omega^n(S^n)$ with $\overline{\operatorname{supp} \omega} \subseteq V$ and $\int_M \omega = 1$.

Then

support in $\cup U_i$

$$\deg F = \int_{S^n} F^* \omega = \sum_{i=1}^{k} \int_{U_i} (F|_{U_i})^* (\omega)$$

$$= \sum_{i=1}^{k} \left(\int_V \omega \right) \begin{pmatrix} +1 \text{ if } F|_{U_i} \text{ pres orient} \\ -1 \text{ if } F|_{U_i} \text{ reverses orient} \end{pmatrix}$$

$$= \sum_{i=1}^{k} \begin{cases} +1 & \text{if } dF_{p_i} \text{ pres orient} \\ -1 & \text{if } dF_{p_i} \text{ reverses orient.} \end{cases}$$

∎

Properties: (a) For $F, G : S^n \to S^n$ have

$$\deg(F \circ G) = (\deg F)(\deg G)$$

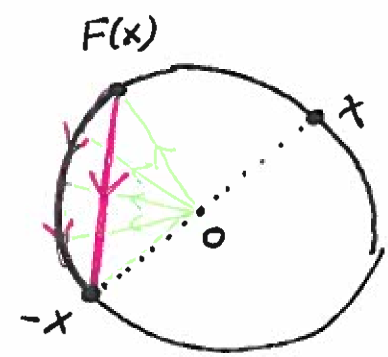
Pf: $(\deg F\circ G)[\omega] = (F \circ G)^*[\omega] = G^*(F^*[\omega])$

$$= G^*((\deg F)[\omega]) = (\deg G)(\deg F)[\omega]$$

(b) If $A : S^n \to S^n$ is the antipodal map $x \mapsto -x$ then $\deg A = (-1)^{n+1}$ since A is the composition of $n+1$ reflections.

(c) If $F : S^n \to S^n$ has no fixed points, then $\deg F = (-1)^{n+1}$

Pf: Since $F(x) \neq x$, the line segment joining $-x$ to $F(x)$ does not go through 0.



So $H : S^n \times I \to S^n$ given by

$$H(x,t) = \frac{(1-t)F(x) - tx}{\|(1-t)F(x) - tx\|}$$

makes sense and shows that F is homotopic to A.

So $\deg F = \deg A = (-1)^{n+1}$.

<u>Thm</u> S^n has a nowhere vanishing vector field iff n is odd

<u>Pf</u>: Suppose $X \in \mathfrak{X}(S^n)$ is nowhere vanishing. Let θ_t be the associated flow. Choose $\varepsilon > 0$ so that θ_ε has no fixed points (can do since $X_p \neq 0$ and S^n is cpt.)

Thus $\deg(\theta_\varepsilon) = (-1)^{n+1}$. As θ_ε is homotopic to id_{S^n} (via θ_t) have $\deg \theta_\varepsilon = \deg \mathrm{id}_{S^n} = 1$. So n is odd.

When n is odd, you constructed nowhere vanishing vector fields on the HW. E.g. view $S^n \subseteq \mathbb{R}^{n+1} \subseteq \mathbb{C}^{\frac{n+1}{2}}$ and consider the flow $\theta_t(z) = e^{it} z$. $\blacksquare$